Answers - Vectors in two dimensions

Year: Year 12

Answer to Question 1

a. Add the $\mathbf{i}$ components and the $\mathbf{j}$ components separately. $(3-1)\mathbf{i}+(-2+5)\mathbf{j}=2\mathbf{i}+3\mathbf{j}$.

b. Multiply each vector by its scalar first, then subtract component by component. $2\mathbf{a}=6\mathbf{i}-4\mathbf{j}$ and $3\mathbf{b}=-3\mathbf{i}+15\mathbf{j}$. So $2\mathbf{a}-3\mathbf{b}=9\mathbf{i}-19\mathbf{j}$. So $2\mathbf{a}-3\mathbf{b}=9\mathbf{i}-19\mathbf{j}$.

c. The magnitude of $p\mathbf{i}+q\mathbf{j}$ is $\sqrt{p^2+q^2}$. $|\mathbf{a}|=\sqrt{3^2+(-2)^2}=\sqrt{13}$. So $|\mathbf{a}|=\sqrt{13}\approx 3.61$.

d. A unit vector has magnitude 1, so divide the vector by its own magnitude. $|\mathbf{b}|=\sqrt{1+25}=\sqrt{26}$. Unit vector $=\dfrac{1}{\sqrt{26}}(-\mathbf{i}+5\mathbf{j})$. Check: its magnitude is $\dfrac{\sqrt{26}}{\sqrt{26}}=1$ ✓.

Answer to Question 2

a. Use Pythagoras for the magnitude; a sketch shows which quadrant the vector is in. $\sqrt{25+144}=$ **13**. The vector is in the second quadrant: $\tan^{-1}\left(\frac{12}{5}\right)=67.4^\circ$ above the negative axis, so the angle is $180^\circ-67.4^\circ=$ **112.6**$^\circ$.

b. This vector points right and down, so it is in the fourth quadrant. $\sqrt{16+9}=$ **5**. $\tan^{-1}\left(\frac{3}{4}\right)=36.9^\circ$ below the positive x-axis, so the angle is $360^\circ-36.9^\circ=$ **323.1**$^\circ$.

c. A vector of magnitude r at angle θ to the positive x-axis is $r\cos\theta\,\mathbf{i}+r\sin\theta\,\mathbf{j}$. $10\cos 30^\circ\mathbf{i}+10\sin 30^\circ\mathbf{j}=5\sqrt{3}\,\mathbf{i}+5\mathbf{j}$. Check: $\sqrt{75+25}=10$ ✓.

Answer to Question 3

a. Parallel vectors are scalar multiples of each other. $\mathbf{p}=m\mathbf{q}$ so $3=-6m$, $m=-\frac{1}{2}$. Then $\lambda=-\frac{1}{2}\times 8=-4$. Check: $3\mathbf{i}-4\mathbf{j}=-\frac{1}{2}(-6\mathbf{i}+8\mathbf{j})$ ✓.

b. Look for a single number that multiplies both components. $\begin{pmatrix}-6\\9\end{pmatrix}=-\frac{3}{2}\begin{pmatrix}4\\-6\end{pmatrix}$ because $-\frac{3}{2}\times 4=-6$ and $-\frac{3}{2}\times(-6)=9$. Parallel, with magnitudes in the ratio $2:3$. The negative scalar shows that the vectors point in opposite directions.

c. For parallel vectors the ratios of corresponding components are equal. Parallel means $\dfrac{2}{k+1}=\dfrac{k}{6}$, so $12=k(k+1)$, $k^2+k-12=0$, $(k+4)(k-3)=0$. $k=3$ **or** $k=-4$. Check $k=3$: $\mathbf{a}=2\mathbf{i}+3\mathbf{j}$ and $\mathbf{b}=4\mathbf{i}+6\mathbf{j}=2\mathbf{a}$ ✓.

Answer to Question 4

a. ; its magnitude comes from Pythagoras.

. $6\sqrt{2}$. So .

b. The midpoint has position vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

$5\mathbf{i}+2\mathbf{j}$. So M is the point $(5, 2)$.

c. $AP : PB = 1 : 2$ means P is one third of the way from A to B.

. $4\mathbf{i}+3\mathbf{j}$. So P is the point $(4, 3)$.

Answer to Question 5

a. Opposite sides of a parallelogram are equal and parallel, so they are represented by the same vector. (since). .

b. The midpoint of a line segment has position vector equal to the mean of the end points' position vectors. Midpoint of OB: $\frac{1}{2}(\mathbf{a}+\mathbf{c})$. Midpoint of AC: $\mathbf{a}+\frac{1}{2}(\mathbf{c}-\mathbf{a})=\frac{1}{2}(\mathbf{a}+\mathbf{c})$. The two expressions are identical.

c. Both midpoints are the same point, so the diagonals of a parallelogram **bisect each other**. This is a general proof, because $\mathbf{a}$ and $\mathbf{c}$ can be any two non-parallel vectors.

Answer to Question 6

a. Speed is the magnitude of the velocity; the bearing is measured clockwise from north. Speed $=\sqrt{9+16}=$ **5 m/s**. Bearing $=\tan^{-1}\left(\frac{3}{4}\right)=$ **036.9°** (east component 3, north 4). The boat moves at 5 m/s on a bearing of $036.9°$.

b. With constant velocity, position = velocity × time. $10\times(3\mathbf{i}+4\mathbf{j})=30\mathbf{i}+40\mathbf{j}$ metres. The boat is 30 m east and 40 m north of the origin, 50 m away.

c. For a bearing θ, the east component is $v\sin\theta$ and the north component is $v\cos\theta$. East: $12\sin 120°=10.39$. North: $12\cos 120°=-6$. Velocity $10.39\mathbf{i}-6\mathbf{j}$ km/h. The negative $\mathbf{j}$ component shows the vessel is heading south of east.

Answer to Question 7

a. Each leg is the difference between consecutive position vectors. Leg 1: $3\mathbf{i}+4\mathbf{j}$. Leg 2: $5\mathbf{i}-3\mathbf{j}$. Leg 3: $0\mathbf{i}-6\mathbf{j}$.

b. Find the magnitude of each leg and add. Lengths: 5, $\sqrt{34} = 5.83$ and 6. Total = **16.83 km**.

c. The straight-line displacement is the sum of the three legs. , length $\sqrt{89}$ = **9.43 km**.

d. Draw a sketch: F is east and south of S, so the bearing is between 090° and 180°. F is 8 km east and 5 km south. The bearing is $90^\circ + \tan^{-1}\left(\frac{5}{8}\right) = 90^\circ + 32.0^\circ = 122^\circ$.

Answer to Question 8

a. The velocity over the ground is the boat's velocity plus the velocity of the current. Velocity $= 3\mathbf{i} + 4\mathbf{j}$, speed 5 m/s. Time $= \dfrac{120}{4}$ = **30 s**. Downstream $= 3 \times 30$ = **90 m**.

b. Measure the bearing clockwise from north using the two components. Bearing $= \tan^{-1}\left(\frac{3}{4}\right)$ = **036.9**$^\circ$.

c. To land opposite, the resultant velocity must have no east component. The eastward component must cancel the current: $4\sin\alpha = 3$, $\sin\alpha = 0.75$, $\alpha \overset{\circ}{=}$ **48.6**.

d. Northward speed $= 4\cos 48.6^\circ = \sqrt{16 - 9} = \sqrt{7} = 2.65$ m/s. Time $= \dfrac{120}{2.646} =$ **45.4 s**. Crossing straight across takes longer, because part of the boat's speed is used against the current.

Answer to Question 9

a. Add the plane's velocity relative to the air and the wind velocity. Ground velocity $= 50\mathbf{i} + 240\mathbf{j}$. Speed $= \sqrt{2500 + 57600}$ = **245.2 km/h**. Bearing $= \tan^{-1}\left(\frac{50}{240}\right)$ = **011.8**$^\circ$.

b. To cancel the eastward wind, the plane must point slightly west of north. The pilot must head west of north: $240\sin\alpha = 50$, $\alpha = 12.0^\circ$. Bearing $\beta = 360^\circ - 12.0^\circ$ = **348.0**$^\circ$.

c. Use Pythagoras: the airspeed is the hypotenuse and the wind is one side. Ground speed $= \sqrt{240^2 - 50^2}$ = **234.7 km/h**.

d. Time $= \dfrac{\text{distance}}{\text{speed}}$; convert the decimal part of an hour to minutes. $\dfrac{600}{234.7} = 2.556$ h = **2 h 33 min**.

Answer to Question 10

a. The velocity is the coefficient of t in each position vector. $\mathbf{v}_P = 3\mathbf{i} + 2\mathbf{j}$ and

$\mathbf{v}_Q = -\mathbf{i} + 4\mathbf{j}$.

b. Substitute $t = 2$ into both position vectors. $t = 2$: $P = (8, 5)$ and $Q = (12, 5)$. Distance $=$ **4 m**.

c. For a collision both coordinates must be equal at the same time. x: $2 + 3t = 14 - t$ gives $t = 3$. y: $1 + 2t = -3 + 4t$ gives $t = 2$. The times differ, so they are never at the same place at the same time.

d. Set the $\mathbf{i}$ components equal. $t = 3$ (at that time $P = (11, 7)$ and $Q = (11, 9)$, 2 m apart).

Answer to Question 11

a. . . So .

b. $AP : PB = 3 : 2$ means P is $\frac{3}{5}$ of the way from A to B. .

$2\mathbf{i}$, i.e. $P = (2, 0)$.

c. Find the magnitude of , remembering that each unit is 10 m.

tens of metres $=$ **141 m**. The wire is about 141 m long.

d. Multiply the length by the cost per metre. $141 \times 48 =$ **£6 768** (using 141 m). The wire costs about £6 768.

Answer to Question 12

a. Each pass vector is the difference between the position vectors.

and .

b. Use Pythagoras for each magnitude. $AB = \sqrt{144 + 81} = 15$ m and $BC = \sqrt{25 + 144} = 13$ m.

c. Time $= \dfrac{\text{distance}}{\text{speed}}$. Total distance 28 m. Time $= \dfrac{28}{15} =$ **1.87 s**.

d. Find and its length, then subtract from the total. , length $\sqrt{49 + 441} = \sqrt{490} = 22.14$ m. Difference $= 28 - 22.14 =$ **5.86 m**.

Answer to Question 13

a. Subtract position vectors. and .

b. Three points are collinear when two of the connecting vectors are parallel and share a point. , so the vectors are parallel and share the point B, so A, B, C lie on a straight line.

c. Since , BC is twice as long as AB. $AB : BC = 1 : 2$.

d. Find and convert units of 100 m to metres. , length $\sqrt{405} = 20.12$ units $= 2\,012$ m. Cost $= 2012 \times 18 =$ **£36 216**.

Answer to Question 14

a. After 2 hours, A has travelled 48 km and B 36 km; resolve using $\sin$ for east and $\cos$ for north. A: $48\sin 60°\mathbf{i} + 48\cos 60°\mathbf{j} = 41.6\mathbf{i} + 24.0\mathbf{j}$. B: $36\sin 150°\mathbf{i} + 36\cos 150°\mathbf{j} = 18.0\mathbf{i} - 31.2\mathbf{j}$.

b. The difference between the bearings is $150° - 60° = 90°$. The angle between the two courses is $90°$, so the distance is $\sqrt{48^2 + 36^2} =$ **60 km**. (Check: , length 60.0.) The ships are 60 km apart.

c. points south and west, so the bearing is between $180°$ and $270°$. points south-west: $180° + \tan^{-1}\left(\frac{23.57}{55.18}\right) = 180° + 23.1° =$ **203.1**°.

Answer to Question 15

a. Opposite sides of a parallelogram give and . , so and . , so and .

b. .

c. is a scalar multiple of , so MN is **parallel** to AC and two thirds of its length.

d. The scale factor is $\frac{2}{3}$ (since $BM = \frac{2}{3}BA$ and $BN = \frac{2}{3}BC$), which matches $MN = \frac{2}{3}AC$.

Answer to Question 16

a. Subtract position vectors: [redacted] . [redacted] and [redacted] .

b. [redacted] . [redacted] : parallel vectors with the common point P, so P, Q, R are collinear.

c. [redacted] , so $PQ : QR = \mathbf{1:1}$; Q is the midpoint of PR.

d. [redacted] must be a multiple of $\mathbf{a} + \mathbf{b}$, so the coefficients of $\mathbf{a}$ and $\mathbf{b}$ must be equal. The line has direction $\mathbf{a} + \mathbf{b}$, so [redacted] , giving $k - 1 = 9$ and $k = 10$. Check: [redacted] continues the pattern $2, 5, 8, 11$ and $1, 4, 7, 10$ ✓.

Answer to Question 17

a. The velocity is the coefficient of t; speed is its magnitude. Velocity $3\mathbf{i} + 2\mathbf{j}$, speed $\sqrt{13} = \mathbf{3.61\ km/h}$.

b. [redacted] . [redacted] .

[redacted] .

This is the required expression.

c. Complete the square to find the minimum of the squared distance. $13t^2 - 52t + 65 = 13(t - 2)^2 + 13$. Minimum when $t = \mathbf{2\ h}$, and the distance is $\sqrt{13} = \mathbf{3.61\ km}$.

d. The boat is on the x-axis when its $\mathbf{j}$ component is zero. $-1 + 2t = 0$ gives $t = \mathbf{0.5\ h}$. The boat crosses the coastline after half an hour, at the point $(3.5, 0)$.

Answer to Question 18

a. For a bearing θ, east $= d \sin\theta$ and north $= d \cos\theta$.
$6 \sin 250^\circ = 6 \times (-0.9397) = -5.64$ and $6 \cos 250^\circ = 6 \times (-0.3420) = -2.05$ ✓.
Totals: east $2.50 + 6.93 - 5.64 = 3.79$, north $4.33 - 4.00 - 2.05 = -1.72$.

b. Add the east components and the north components. Displacement $= 3.79\mathbf{i} - 1.72\mathbf{j}$ km.

c. Use Pythagoras. Distance $= \sqrt{3.79^2 + 1.72^2} = \sqrt{17.32} = \mathbf{4.16\ km}$.

d. The hiker is south-east of the start. Bearing of hiker from start

$= 90^\circ + \tan^{-1}\left(\frac{1.72}{3.79}\right) = 114.4^\circ$. The return bearing is $114.4^\circ + 180^\circ =$ **294.4**$^\circ$.

Answer to Question 19

a. Each midpoint is the mean of the two end points. $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\mathbf{r} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$, $\mathbf{s} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$.

b. .

c. , so PQ and SR are equal and parallel, so $PQRS$ is a **parallelogram**.

d. Similarly . $PQ = \frac{1}{2}AC$ and $QR = \frac{1}{2}BD$. In a rectangle the diagonals AC and BD are equal, so $PQ = QR$ and $PQRS$ is a **rhombus**.

Answer to Question 20

a. Position = initial position + velocity $\times t$. $\mathbf{r}_P = 200t\mathbf{i} + 150t\mathbf{j}$ and $\mathbf{r}_Q = (300 - 100t)\mathbf{i} + 140t\mathbf{j}$.

b. .

c. $|PQ|^2 = (300 - 300t)^2 + 100t^2 = 90000 - 180000t + 90000t^2 + 100t^2 = 90100t^2 - 180000t + 90000$. The minimum of $at^2 + bt + c$ occurs at $t = -\dfrac{b}{2a} = \dfrac{180000}{180200} = 0.9989$ h. Minimum value $= 90000 - \dfrac{180000^2}{4 \times 90100} = 99.9$, so the minimum distance is $\sqrt{99.9} \approx$ **10 km**.

d. 10 km is less than 15 km, so the separation rule **is broken** (at about 59.9 minutes).

Answer to Question 21

a. Position = initial position + velocity $\times t$. $(12 - 3t)\mathbf{i} + (16 - 2t)\mathbf{j}$. This gives the yacht's position t hours after 12:00.

b. The lifeboat travels $10T$ km in a straight line, so the distance to the interception point equals $10T$: $(12 - 3T)^2 + (16 - 2T)^2 = 100T^2$. Expand: $144 - 72T + 9T^2 + 256 - 64T + 4T^2 = 100T^2$, so $87T^2 + 136T - 400 = 0$. So $a = 87$, $b = 136$ and $c = -400$.

c. Use the quadratic formula and keep the positive root, since time cannot be negative.

$$T = \frac{-136 + \sqrt{136^2 + 4 \times 87 \times 400}}{2 \times 87} = \frac{-136 + \sqrt{157696}}{174} = \frac{-136 + 397.1}{174} = 1.50$$

h. Interception at about 13:30. Yacht position: $(12 - 4.50,\ 16 - 3.00) = (7.5, 13)$ km. Check: the distance from H is $\sqrt{7.5^2 + 13^2} = 15.0$ km $= 10 \times 1.5$ ✓.