Lesson - Trigonometry

Year: Year 12

Year 12 trigonometry has three strands: solving any triangle, working in radians, and using graphs and identities to solve trig equations. Learn the exact values and the CAST idea, and always check whether the question wants degrees or radians.

Starter activity

A sailor leaves a harbour and travels 10 km due east and then 6 km on a bearing of 030°. She wants to sail home in a straight line. Sketch her route as a triangle and mark the angle at the turning point. How far is the journey home, and in what direction must she steer? Think about which rule could help if you know two sides and the angle between them, and which rule you could use afterwards to find the direction.

Sine rule, cosine rule and area

Rule: sine rule, cosine rule and area

For a triangle with sides a, b, c opposite angles A, B, C:

- Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (a side and its opposite angle are known).
- Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$, or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (two sides and the angle between them, or three sides).
- Area $= \frac{1}{2}ab\sin C$.

Worked example 1

In triangle ABC, $AB = 9$ cm, $AC = 12$ cm and angle $A = 40^\circ$. Find BC and the area.

By the cosine rule $BC^2 = 81 + 144 - 2(9)(12)\cos 40^\circ = 59.5$, so $BC = 7.72$ cm. The area is $\frac{1}{2}(9)(12)\sin 40^\circ = 34.7$ cm^2.

Watch out: the ambiguous case

If you are given two sides and a non-included angle there may be two triangles, because $\sin\theta = \sin(180^\circ - \theta)$.

Worked example 2

In triangle ABC, $a = 8$, $b = 10$ and $A = 30^\circ$. Find B.

$\sin B = \dfrac{10\sin 30^\circ}{8} = 0.625$, so $B = 38.7^\circ$ or $B = 180^\circ - 38.7^\circ = 141.3^\circ$. Both are valid because $30^\circ + 141.3^\circ$: there are two possible triangles.

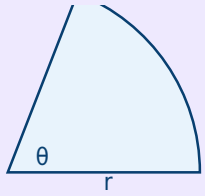
Radians

Definition: radian

One **radian** is the angle at the centre of a circle subtended by an arc equal in length to the radius. π radians $= 180^\circ$. To convert degrees to radians multiply by $\dfrac{\pi}{180}$; to convert back multiply by $\dfrac{180}{\pi}$. So $150^\circ = \dfrac{5\pi}{6}$ and $\dfrac{3\pi}{4} = 135^\circ$.

Rule: arcs and sectors (θ in radians)

Arc length $l = r\theta$, sector area $A = \frac{1}{2}r^2\theta$, segment area $= \frac{1}{2}r^2(\theta - \sin\theta)$.



Worked example 3

A sector has radius 8 cm and area 24 cm^2. Find the angle and the arc length.

$24 = \frac{1}{2}(64)\theta$, so $\theta = 0.75$ rad and the arc length is $8 \times 0.75 = 6$ cm.

Exact values

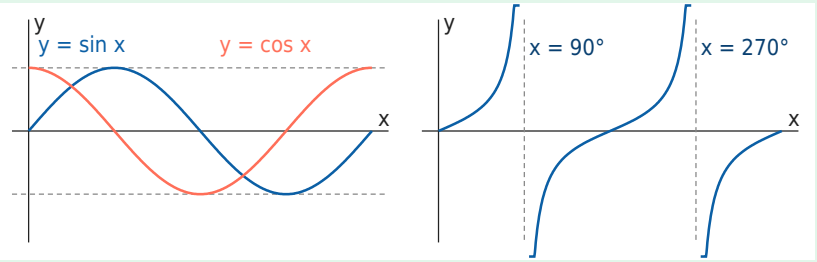
Key fact: exact values

θ	0	30° $(\frac{\pi}{6})$	45° $(\frac{\pi}{4})$	60° $(\frac{\pi}{3})$	90° $(\frac{\pi}{2})$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Graphs of trig functions

Key fact: sine, cosine and tangent graphs

$y = \sin x$ and $y = \cos x$ have period 360° (2π), amplitude 1, and lie between -1 and 1. $y = \tan x$ has period 180° (π) and vertical asymptotes at $x = 90^\circ, 270^\circ, \ldots$. For $y = a\sin bx$ the amplitude is a and the period is $\dfrac{360^\circ}{b}$.



Identities

Rule: two identities

$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$ (Pythagoras in a right-angled triangle with hypotenuse 1). Rearranged: $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$.

Worked example 4

Prove that $\dfrac{\cos\theta}{1-\sin\theta} \equiv \dfrac{1+\sin\theta}{\cos\theta}$.

Multiply the left side by $\dfrac{1+\sin\theta}{1+\sin\theta}$: $\dfrac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} = \dfrac{\cos\theta(1+\sin\theta)}{\cos^2\theta} = \dfrac{1+\sin\theta}{\cos\theta}$. ■

Watch out

Work on one side only when proving an identity, and state the identity you use. Never work on both sides at once.

Solving trig equations

Method: solving $\sin\theta = k$, $\cos\theta = k$ or $\tan\theta = k$

1. Rearrange to $\sin\theta = k$, $\cos\theta = k$ or $\tan\theta = k$ (use identities to get a quadratic in one function first).
2. Find the principal value α with your calculator.
3. Use the graph or CAST to find the other solutions in the range: $\sin\theta$: $180^\circ - \alpha$; $\cos\theta$: $360^\circ - \alpha$; $\tan\theta$: $\alpha + 180^\circ$.
4. For $\sin 2\theta$, solve for 2θ over the doubled range, then halve.

Worked example 5

Solve $2\cos^2\theta + \sin\theta - 1 = 0$ for $0 \leq \theta \leq 360°$.

Replace $\cos^2\theta$ by $1 - \sin^2\theta$: $2 - 2\sin^2\theta + \sin\theta - 1 = 0$, so $2\sin^2\theta - \sin\theta - 1 = 0$ and $(2\sin\theta + 1)(\sin\theta - 1) = 0$. Then $\sin\theta = 1$ gives $\theta = 90°$; $\sin\theta = -\frac{1}{2}$ gives $\theta = 210°, 330°$.

Worked example 6

Solve $\tan\theta = -1$ for $0 \leq \theta \leq 360°$.

The calculator gives $-45°$, outside the range. Add $180°$ to get $135°$, then another $180°$ to get $315°$. So $\theta = 135°$ or $315°$.

Watch out

Check the calculator mode (degrees or radians) and never use $l = r\theta$ with θ in degrees. Do not cancel $\sin\theta$ from both sides of an equation: you lose solutions. Do not stop at the calculator answer: find all the solutions in the range.

Worked example 7: The sine rule and two possible angles

In triangle PQR, $PQ = 8$ cm, $QR = 6$ cm and angle $P = 35°$. Find the possible values of angle R.

By the sine rule $\dfrac{\sin R}{8} = \dfrac{\sin 35°}{6}$, so $\sin R = \dfrac{8\sin 35°}{6} = 0.7648$. Then $R = 49.9°$ or $R = 180° - 49.9° = 130.1°$. Check whether each fits in the triangle: with $R = 130.1°$ the angle sum is $35° + 130.1° = 165.1°$, so it is valid; and $35° + 49.9° = 84.9°$ is also valid. Both triangles exist because the side opposite the given angle (6 cm) is shorter than the other given side (8 cm). This is the ambiguous case of the sine rule.

Worked example 8: Using the cosine rule to find an angle

A triangle has sides 7 cm, 9 cm and 12 cm. Find all three angles.

The largest angle is opposite 12 cm: $\cos C = \dfrac{7^2 + 9^2 - 12^2}{2 \times 7 \times 9} = \dfrac{-14}{126} = -0.1111$, so $C = 96.4°$. The angle opposite 9 cm: $\cos B = \dfrac{7^2 + 12^2 - 9^2}{2 \times 7 \times 12} = \dfrac{112}{168} = 0.6667$, so $B = 48.2°$. Then $A = 180° - 96.4° - 48.2° = 35.4°$. Check with the cosine rule for A: $\cos A = \dfrac{9^2 + 12^2 - 7^2}{2 \times 9 \times 12} = \dfrac{176}{216} = 0.8148$, giving $35.4°$ ✓.

Worked example 9: Radians and arc length in a context

A pendulum of length 80 cm swings through an angle of 0.35 radians. Find the length of the arc traced by the bob and the area swept out by the string.

Arc length $s = r\theta = 80 \times 0.35 = 28$ cm. Area swept $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 6400 \times 0.35 = 1\,120$ cm^2. In degrees the swing is $0.35 \times \dfrac{180}{\pi} = 20.05^\circ$, so about 20°. Using radians makes the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ much simpler than the degree versions $\dfrac{\theta}{360} \times 2\pi r$ and $\dfrac{\theta}{360} \times \pi r^2$.

Worked example 10: Exact values from special triangles

Use an equilateral triangle of side 2 to show that $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$, and use a right-angled isosceles triangle to find $\tan 45^\circ$.

Draw an equilateral triangle with side 2 and split it down the middle. This gives a right-angled triangle with hypotenuse 2, shortest side 1 and height $\sqrt{4-1} = \sqrt{3}$. The angle at the apex is 30°, so $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$, and for the angle 60°, $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$ and $\tan 60^\circ = \sqrt{3}$. A right-angled isosceles triangle with equal sides 1 has hypotenuse $\sqrt{2}$, so $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$ and $\tan 45^\circ = 1$.

Worked example 11: Solving with a multiple angle in radians

Solve $2\cos\left(3x - \dfrac{\pi}{4}\right) = \sqrt{2}$ for $0 \le x \le \pi$.

$\cos\left(3x - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$. Let $u = 3x - \dfrac{\pi}{4}$; as x runs from 0 to π, u runs from $-\dfrac{\pi}{4}$ to $\dfrac{11\pi}{4}$.

$\cos u = \dfrac{\sqrt{2}}{2}$ when $u = \pm\dfrac{\pi}{4} + 2n\pi$. The values in range are $u = -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{7\pi}{4}, \dfrac{9\pi}{4}$.

Then $3x = u + \dfrac{\pi}{4}$ gives $3x = 0, \dfrac{\pi}{2}, 2\pi, \dfrac{5\pi}{2}$, so $x = 0, \dfrac{\pi}{6}, \dfrac{2\pi}{3}, \dfrac{5\pi}{6}$.

Worked example 12: Proving an identity by working on one side

Prove that $(\sin\theta + \cos\theta)^2 \equiv 1 + 2\sin\theta\cos\theta$ and use it to find $\sin\theta\cos\theta$ when $\sin\theta + \cos\theta = \frac{7}{5}$.

Expand the bracket: $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$ using $\sin^2\theta + \cos^2\theta = 1$. If $\sin\theta + \cos\theta = \frac{7}{5}$ then $\frac{49}{25} = 1 + 2\sin\theta\cos\theta$, so $2\sin\theta\cos\theta = \frac{24}{25}$ and $\sin\theta\cos\theta = \frac{12}{25}$. These values agree with $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$ since $\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$ and $\frac{3}{5} \times \frac{4}{5} = \frac{12}{25}$ ✓.

Worked example 13: A graph transformation question in trigonometry

Sketch $y = 2\cos(x - 60^\circ)$ for $0^\circ \le x \le 360^\circ$, stating the coordinates of the maximum, the minimum and the y-intercept.

Start with $y = \cos x$, which has a maximum at $(0^\circ, 1)$. The factor 2 stretches the graph vertically so the amplitude is 2. Replacing x by $x - 60^\circ$ translates it 60° to the right. So the maximum is at $(60^\circ, 2)$ and a second maximum would be at 420°, outside the range. The minimum is at $(240^\circ, -2)$. The y-intercept is $2\cos(-60^\circ) = 2 \times \frac{1}{2} = 1$, so $(0, 1)$. The curve crosses the x-axis where $x - 60^\circ = 90^\circ$ and 270°, that is at $x = 150^\circ$ and $x = 330^\circ$.

Worked example 14: Using the three area formulae together

In triangle ABC the area is 30 cm^2, $AB = 10$ cm and $AC = 8$ cm. Angle A is acute. Find angle A and the length of BC.

Area $= \frac{1}{2} \times 10 \times 8 \sin A = 40 \sin A = 30$, so $\sin A = 0.75$ and $A = 48.6^\circ$ (acute). Now the cosine rule: $BC^2 = 10^2 + 8^2 - 2 \times 10 \times 8 \cos 48.6^\circ = 164 - 160 \times 0.6614 = 58.2$, so $BC = 7.63$ cm. A useful check is the converse: the area with sides 10, 8 and included angle 48.6° is $40 \sin 48.6^\circ = 30.0$ cm^2 ✓. If the question had allowed an obtuse angle, the second solution would have been $A = 131.4^\circ$ with $BC^2 = 164 + 160 \times 0.6614 = 269.8$ and $BC = 16.4$ cm.

Note: choosing between degrees and radians

Degrees are convenient for bearings and for angles in diagrams of triangles. Radians are used whenever arc lengths, sector areas, small-angle approximations or calculus appear: the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ only hold when θ is in radians. Read the question carefully. If an interval is given as $0 \le x \le 2\pi$ your answers must be in radians, preferably exact multiples of π such as $\frac{\pi}{6}$ or $\frac{5\pi}{3}$. Before every trigonometry calculation check the mode of the calculator: a wrong mode is the most common source of lost marks.

Method: solving a trigonometric equation

1. Rewrite the equation so that a single trigonometric function appears (use $\sin^2\theta + \cos^2\theta = 1$ or $\tan\theta = \sin\theta / \cos\theta$ if necessary).
2. If it is a quadratic in $\sin\theta$, $\cos\theta$ or $\tan\theta$, factorise and solve each factor; reject values outside -1 to 1 for sine and cosine.
3. Find the principal value on a calculator and use the symmetry of the graph (or a CAST diagram) to find all the solutions in the range.
4. For $\sin(n\theta)$ and similar, widen the range first, solve, then divide by n.
5. Never divide through by $\sin\theta$ or $\cos\theta$ unless you check separately the case where it is zero.

Watch out: common trigonometry mistakes

- Having the calculator in degrees when the question uses radians, or vice versa.
- Using the sine rule in the ambiguous case without checking whether $180^\circ - \theta$ also works.
- Writing $\sin^2\theta$ as $\sin\theta^2$ or $2\sin\theta$ for $\sin 2\theta$. They are different.
- Dividing by $\sin\theta$ or $\cos\theta$ when solving, which loses solutions.
- Rounding too early; keep full calculator values until the final answer.

Did you know?

The word *sine* comes from a mistranslation. Indian mathematicians called the half-chord *jya*; Arab scholars wrote it as *jiba*, and when this was translated into Latin in the 12th century, it was confused with *jaib* (a fold or bay) and became *sinus*. Radians were first named by James Thomson in 1873, but the idea of measuring angles by arc length goes back to Roger Cotes in 1714. GPS and satellite navigation use the sine and cosine rules constantly.

Otto's tip

Sketch the graph first. The picture tells you how many solutions to expect in the range. Keep full calculator values until the last line and give 3 significant figures.