Answers - Trigonometry

Year: Year 12

Answer to Question 1

a. The angles of a triangle add up to 180°. $C = 180^\circ - 48^\circ - 65^\circ = 67^\circ$.

b. Side b is opposite angle B, and we know the pair a and A; use $\frac{a}{\sin A} = \frac{b}{\sin B}$. $\frac{b}{\sin 65^\circ} = \frac{12}{\sin 48^\circ}$, so $b = \frac{12 \sin 65^\circ}{\sin 48^\circ} =$ **14.6 cm**. So $b \approx 14.6$ cm.

c. Use the sine rule again with the pair c and $C = 67^\circ$. $c = \frac{12 \sin 67^\circ}{\sin 48^\circ} =$ **14.9 cm**. Check: the longest side c is opposite the largest angle C ✓.

Answer to Question 2

a. Two sides and the included angle are known, so use $c^2 = a^2 + b^2 - 2ab\cos C$. $c^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \cos 52^\circ = 130 - 126 \cos 52^\circ = 52.43$, so $c =$ **7.24 cm**.

b. Three sides are known, so rearrange the cosine rule to find an angle. The smallest angle is opposite the shortest side (8 cm): $\cos A = \frac{11^2 + 14^2 - 8^2}{2 \times 11 \times 14} = \frac{253}{308}$, so $A =$ **34.8°**.

c. Use area $= \frac{1}{2}ab\sin C$ with the included angle. Area $= \frac{1}{2} \times 6.4 \times 9.1 \times \sin 38^\circ =$ **17.9 cm²**.

d. Since $\sin(180^\circ - C) = \sin C$, there are two angles with the same sine. $\frac{1}{2} \times 8 \times 10 \sin C = 20$, so $\sin C = 0.5$ and $C = 30^\circ$ **or** 150°. Both triangles have area 20 cm².

Answer to Question 3

a. Multiply degrees by $\frac{\pi}{180}$ to get radians, and multiply radians by $\frac{180}{\pi}$ to get degrees. $135^\circ = 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$. $\frac{5\pi}{6} = \frac{5 \times 180^\circ}{6} = 150^\circ$.

b. In radians, arc length $= r\theta$ and sector area $= \frac{1}{2}r^2\theta$. Arc $= r\theta = 8 \times 1.2 =$ **9.6 cm**. Area $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 64 \times 1.2 =$ **38.4 cm²**.

c. Rearrange $s = r\theta$ to find θ, then use the area formula. $\theta = \frac{s}{r} = \frac{15}{6} =$ **2.5 rad**. Area $= \frac{1}{2} \times 36 \times 2.5 =$ **45 cm²**.

d. Segment area = sector area − triangle area $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$; the calculator must be in radians. Segment

$= \frac{1}{2}r^2(\theta - \sin\theta) = 50(1.2 - \sin 1.2) = 50(1.2 - 0.9320) =$ **13.40 cm**^{2}. The segment has area 13.40 cm^2.

Answer to Question 4

a. Use the exact values $\sin 60^\circ = \cos 30^\circ = \dfrac{\sqrt{3}}{2}$ and $\tan 45^\circ = 1$.

$\dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{3}}{2} - 1 = \dfrac{3}{4} - 1 = -\dfrac{1}{4}$. So the exact value is $-\frac{1}{4}$.

b. Use $\tan\frac{\pi}{3} = \sqrt{3}$ and $\sin\frac{\pi}{4} = \dfrac{\sqrt{2}}{2}$; dividing by a fraction means multiplying by its reciprocal. $\dfrac{\sqrt{3}}{\sqrt{2}/2} = \dfrac{2\sqrt{3}}{\sqrt{2}} = \sqrt{2} \times \sqrt{3} = \sqrt{6}$. Note that $\dfrac{2}{\sqrt{2}} = \sqrt{2}$.

c. Use the related acute angle and the CAST diagram: 150° is in the second quadrant, where only sine is positive; 210° is in the third quadrant, where only tangent is positive; 135° is in the second quadrant. $\cos 150^\circ = -\cos 30^\circ = -\dfrac{\sqrt{3}}{2}$. $\sin 210^\circ = -\sin 30^\circ = -\dfrac{1}{2}$. $\tan 135^\circ = -\tan 45^\circ = -1$. These signs agree with the CAST diagram.

Answer to Question 5

a. Use $\sin^2\theta + \cos^2\theta \equiv 1$ to replace the numerator, then cancel $\sin\theta$. LHS $= \dfrac{\sin^2\theta}{\sin\theta\cos\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta =$ RHS. This proves the identity.

b. Write $\tan\theta$ as $\dfrac{\sin\theta}{\cos\theta}$, then add the fractions over a common denominator. LHS $= \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta\cos\theta} =$ RHS. This proves the identity.

c. Use $\cos^2\theta = 1 - \sin^2\theta$; an acute angle has a positive cosine. $\cos^2\theta = 1 - \dfrac{9}{25} = \dfrac{16}{25}$, and θ is acute so $\cos\theta = \dfrac{4}{5}$. $\tan\theta = \dfrac{3/5}{4/5} = \dfrac{3}{4}$. This is the 3-4-5 right-angled triangle.

Answer to Question 6

a. Use $\sin^{-1}$ for the principal value; sine is also positive in the second quadrant, giving $180^\circ - \alpha$. Principal value 36.9°; the second solution is $180^\circ - 36.9^\circ = 143.1^\circ$. So $\theta = 36.9^\circ$ **and** 143.1°.

b. Divide by 2 first; cosine is negative in the second and third quadrants, with related angle 60°. $\cos\theta = -\frac{1}{2}$: 120° **and** 240°.
c. Tangent has period 180°, so add multiples of 180° to the principal value. Principal value -63.4°, add 180° and 360°: 116.6° **and** 296.6°.
d. Double the interval, because θ from 0° to 360° means 2θ from 0° to 720°. Let $u = 2\theta$ in the range $0^\circ \le u \le 720^\circ$. $\sin u = 0.5$ gives $u = 30^\circ, 150^\circ, 390^\circ, 510^\circ$, so $\theta = 15^\circ, 75^\circ, 195^\circ, 255^\circ$.

Answer to Question 7

a. Two sides and the included angle are known, so use the cosine rule. $AB^2 = 120^2 + 95^2 - 2 \times 120 \times 95\cos 70^\circ = 23425 - 22800\cos 70^\circ = 15627$, so $AB =$ **125 m**. The lake is about 125 m wide.
b. Use area $= \frac{1}{2}ab\sin C$. Area $= \frac{1}{2} \times 120 \times 95 \times \sin 70^\circ =$ **5 356 m^2**. The area is about 5 356 m^2.
c. Multiply the area by the price per square metre. $5356 \times 14.50 = 77\,662$, so about **£77 700**. The land costs about £77 700.
d. Use the sine rule with the side $OB = 95$ m opposite angle OAB. $\dfrac{\sin A}{95} = \dfrac{\sin 70^\circ}{125.0}$, so $\sin A = 0.7141$ and $A =$ **45.6**$^\circ$. Angle $OAB \approx 45.6^\circ$.

Answer to Question 8

a. Draw north lines at P and at Q; the bearing of P from Q is the back-bearing. At Q, the back-bearing to P is $040^\circ + 180^\circ = 220^\circ$. The ship\'s new direction is 110°. The angle between them is $220^\circ - 110^\circ = 110^\circ$.
b. Use the cosine rule with $PQ = 20$, $QR = 30$ and the included angle 110°. $PR^2 = 20^2 + 30^2 - 2 \times 20 \times 30\cos 110^\circ = 1300 + 410.4 = 1710.4$, so $PR =$ **41.4 km**.
c. Find angle QPR with the sine rule, then add it to the bearing of Q from P. $\dfrac{\sin P}{30} = \dfrac{\sin 110^\circ}{41.36}$, so $\sin P = 0.6816$ and angle $QPR = 43.0^\circ$. Bearing $= 040^\circ + 43^\circ = 083^\circ$.
d. Time $= \dfrac{\text{distance}}{\text{speed}}$. Time $= \dfrac{41.36}{18} = 2.298$ h $= 2$ h 18 min (to the nearest minute).

Answer to Question 9

a. Use angles on a straight line at B and then the angle sum of triangle ATB. Angle TBA is $180^\circ - 41^\circ = 139^\circ$ (angles on a straight line). Angle $ATB = 180^\circ - 32^\circ - 139^\circ = 9^\circ$.
b. Use the sine rule: side AT is opposite angle $TBA = 139^\circ$, and side $AB = 40$ m is

opposite angle $ATB = 9°$. $\frac{AT}{\sin 139°} = \frac{40}{\sin 9°}$, so $AT = \frac{40 \sin 139°}{\sin 9°} =$ **167.8 m**. So $AT \approx 167.8$ m.

c. In the right-angled triangle formed by A, the top and the base of the tower, the height is opposite the $32°$ angle. Height $= AT \sin 32° = 167.75 \times 0.5299 =$ **89 m**. The tower is about 89 m tall.

d. In the right-angled triangle at B, $\tan 41° = \frac{\text{height}}{BH}$. Height $= BH \tan 41°$ where BH is the distance, so $BH = \frac{88.9}{\tan 41°} =$ **102 m**. So B is about 102 m from the base of the tower.

Answer to Question 10

a. Substitute $t = 3$ and $t = 9$ into the model. $t = 3$: $4 + 2 \sin 90° =$ **6 m**. $t = 9$: $4 + 2 \sin 270° =$ **2 m**. These are the high and low tide depths.

b. Set $h = 5$ and solve for $30t$ in the range $0°$ to $360°$. $2 \sin(30t) = 1$, $\sin(30t) = 0.5$, $30t = 30°$ or $150°$, so $t =$ **1 and 5**: 1 am and 5 am.

c. Solve $4 + 2 \sin(30t) \geq 4.5$; the sine is at least 0.25 between the two solutions in the first half-cycle. $\sin(30t) \geq 0.25$: $30t \geq 14.48°$ and $30t \leq 165.52°$, so $0.483 \leq t \leq 5.517$. That is from **12:29 am to 5:31 am**.

d. The period of $\sin(kx°)$ is $\frac{360°}{k}$. Period $= \frac{360°}{30°} = 12$ hours: the tide pattern repeats every 12 hours (a simplification of the real 12.4 hours).

Answer to Question 11

a. In radians, the area of a sector is $\frac{1}{2} r^2 \theta$. Area $= \frac{1}{2} r^2 \theta = \frac{1}{2} \times 36 \times \frac{2\pi}{3} = 12\pi =$ **37.7 m**^{2}. The area is exactly 12π m^2.

b. The arc length is $r\theta$; add the two radii for the straight edges. Arc $= r\theta = 6 \times \frac{2\pi}{3} = 4\pi = 12.57$ m. Perimeter $= 4\pi + 6 + 6 =$ **24.6 m**. The perimeter is about 24.6 m.

c. Multiply the perimeter by the cost per metre. Cost $= 24.57 \times 6.50 =$ **£159.68**. The fencing costs about £159.68.

d. Multiply the area by the cost per square metre. Cost $= 37.70 \times 4.20 =$ **£158.34**. The turf costs about £158.34.

Answer to Question 12

a. Split the isosceles triangle OAB into two right-angled triangles, each with angle

0.6 rad at O; the calculator must be in radians. $AB = 2r\sin\dfrac{\theta}{2} = 20\sin 0.6 =$ **11.29 cm**. The chord is about 11.29 cm long.

b. Use area $= \frac{1}{2}r^2\sin\theta$ with the angle between the two radii. Triangle $= \frac{1}{2}r^2\sin\theta = 50\sin 1.2 = 46.60$ cm^2. The triangle has area about 46.60 cm^2.

c. The segment is the sector minus the triangle. Sector $= \frac{1}{2}\times 100\times 1.2 = 60$ cm^2. Segment $= 60 - 46.60 =$ **13.40 cm**^{2}. The segment has area about 13.40 cm^2.

d. Multiply the area by the price per cm^2. Cost $= 13.40\times 0.12 =$ **£1.61**. The coloured glass costs about £1.61.

Answer to Question 13

a. The largest angle is opposite the longest side; rearrange the cosine rule. The largest angle is opposite the longest side: $\cos C = \dfrac{120^2 + 150^2 - 200^2}{2\times 120\times 150} = -0.0861$, so $C =$ **94.9**°. The negative cosine shows that the angle is obtuse.

b. Use area $= \frac{1}{2}ab\sin C$ with the two sides enclosing the angle C. Area $= \frac{1}{2}\times 120\times 150\sin 94.94^\circ =$ **8 967 m**^{2}. The field has an area of about 8 967 m^2.

c. Heron's formula is $\sqrt{s(s-a)(s-b)(s-c)}$ where s is half the perimeter. $s = \dfrac{120+150+200}{2} = 235$. Area $= \sqrt{235\times 115\times 85\times 35} = \sqrt{80\,393\,125} = 8\,967$ m^2 ✓. The two methods agree.

d. Multiply the area by the price per square metre. Cost $= 8967\times 0.45 = £4\,035$, i.e. **£4 040**. The seed costs about £4 040.

Answer to Question 14

a. Treat $\sin x$ as a single variable and factorise the quadratic. Let $s = \sin x$: $2s^2 + s - 1 = (2s-1)(s+1)$.

b. Set each factor equal to zero and solve for x in the given range. $\sin x = \frac{1}{2}$ gives $x = 30^\circ$ or 150°. $\sin x = -1$ gives $x = 270^\circ$. Solutions: $30^\circ, 150^\circ, 270^\circ$.

c. $\sin x$ can only take values from -1 to 1, so find the greatest and least values of the quadratic on this interval. $y = 2s^2 + s - 1$ with $-1 \le s \le 1$. At $s = 1$: $y = 2$ (maximum). The quadratic has its minimum at $s = -\frac{1}{4}$: $y = 2\times\frac{1}{16} - \frac{1}{4} - 1 = -\frac{9}{8}$ (minimum). Check the ends: at $s = -1$, $y = 0$. So maximum **2**, minimum $-\frac{9}{8}$.

Answer to Question 15

a. In triangle ABC two sides and the included angle are known, so use the cosine rule. $AC^2 = 8^2 + 6^2 - 2\times 8\times 6\cos 75^\circ = 100 - 96\cos 75^\circ = 75.15$, so $AC =$ **8.67 m**. The diagonal AC is about 8.67 m.

b. In triangle ACD all three sides are now known, so rearrange the cosine rule. $\cos D = \dfrac{9^2 + 5^2 - 75.15}{2 \times 9 \times 5} = \dfrac{30.85}{90} = 0.3427$, so $D =$ **70.0°**. Angle $ADC \approx 70.0°$.

c. Find the area of each triangle with $\frac{1}{2}ab\sin C$ and add. Triangle ABC: $\frac{1}{2} \times 8 \times 6 \sin 75° = 23.18$. Triangle ACD: $\frac{1}{2} \times 9 \times 5 \sin 69.96° = 21.14$. Total = **44.3 m^2**. The plot has a total area of about 44.3 m^2.

d. Multiply the length of AC by the cost per metre. $8.67 \times 11 =$ **£95.37**. The fence costs about £95.37.

Answer to Question 16

a. An equation with both $\sin$ and $\cos^2$ can be written entirely in $\sin\theta$. Replace $\cos^2\theta = 1 - \sin^2\theta$: $2 - 2\sin^2\theta + 5\sin\theta - 4 = 0$, so $2\sin^2\theta - 5\sin\theta + 2 = 0$, $(2\sin\theta - 1)(\sin\theta - 2) = 0$. $\sin\theta = 2$ is impossible. $\sin\theta = \frac{1}{2}$ gives $\theta = 30°, 150°$.

b. Write $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and multiply by $\cos\theta$. $\dfrac{\sin\theta}{\cos\theta} = 2\sin\theta$. Do **not** divide by $\sin\theta$ (it would lose solutions). $\sin\theta = 2\sin\theta\cos\theta$, so $\sin\theta(1 - 2\cos\theta) = 0$. $\sin\theta = 0$: $0°, 180°, 360°$. $\cos\theta = \frac{1}{2}$: $60°, 300°$. Solutions: $0°, 60°, 180°, 300°, 360°$.

c. Shift the interval to match the substitution. Let $u = \theta + 40°$ with $40° \le u \le 400°$. $\cos u = -\frac{1}{2}$ gives $u = 120°, 240°$. So $\theta = 80°, 200°$.

Answer to Question 17

a. The perimeter of a sector is two radii plus the arc. Perimeter $= 2r + r\theta = 20$, so $r\theta = 20 - 2r$ and $\theta = \dfrac{20 - 2r}{r}$.

b. Substitute into $\theta = \dfrac{20 - 2r}{r}$ and $A = \frac{1}{2}r^2\theta$. $r = 4$: $\theta = \dfrac{12}{4} = 3$, area $= \frac{1}{2} \times 16 \times 3 = 24$ ✓. $r = 8$: $\theta = \dfrac{4}{8} = 0.5$, area $= \frac{1}{2} \times 64 \times 0.5 = 16$ ✓.

c. Use $A = \frac{1}{2}r^2\theta$ and substitute $r\theta = 20 - 2r$.
$A = \frac{1}{2}r^2\theta = \frac{1}{2}r(r\theta) = \frac{1}{2}r(20 - 2r) = r(10 - r)$. This is the required result.

d. $A = 10r - r^2 = 25 - (r - 5)^2$. Greatest area **25 cm^2** at $r = 5$, where $\theta = \dfrac{10}{5} =$ **2 rad**. Check: perimeter $= 2 \times 5 + 5 \times 2 = 20$ ✓.

Answer to Question 18

a. $\sin 2x$ varies between -1 and 1, so $3\sin 2x$ varies between -3 and 3. Maximum $3 + 1 =$ **4**, minimum $-3 + 1 = -2$.

b. The period of $\sin kx$ is $\dfrac{360°}{k}$. Period $= \dfrac{360°}{2} = 180°$.

c. Rearrange to make $\sin 2x$ the subject, then solve for $2x$ in $0° \leq 2x \leq 720°$. $3\sin 2x = 1.5$, $\sin 2x = 0.5$. $2x = 30°, 150°, 390°, 510°$, so $x = 15°, 75°, 195°, 255°$ (the points where the red line crosses the curve).

d. A stretch with scale factor $\frac{1}{2}$ parallel to the x-axis, a stretch with scale factor 3 parallel to the y-axis and a translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Answer to Question 19

a. Use the difference of two squares $(1 - \sin\theta)(1 + \sin\theta) = 1 - \sin^2\theta = \cos^2\theta$. Multiply top and bottom of the left side by $(1 + \sin\theta)$: $\frac{\cos\theta(1 + \sin\theta)}{1 - \sin^2\theta} = \frac{\cos\theta(1 + \sin\theta)}{\cos^2\theta} = \frac{1 + \sin\theta}{\cos\theta}$ = RHS. This proves the identity.

b. Add the fractions over the common denominator $(1 + \sin\theta)(1 - \sin\theta)$. LHS $= \frac{(1 - \sin\theta) + (1 + \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)} = \frac{2}{1 - \sin^2\theta} = \frac{2}{\cos^2\theta}$ = RHS. This proves the identity.

c. Use the identity from part (b) to replace the left-hand side. $\frac{2}{\cos^2\theta} = 8$, so $\cos^2\theta = \frac{1}{4}$ and $\cos\theta = \pm\frac{1}{2}$. $\cos\theta = \frac{1}{2}$: $60°, 300°$. $\cos\theta = -\frac{1}{2}$: $120°, 240°$. Solutions: $60°, 120°, 240°, 300°$.

Answer to Question 20

a. Use the sine rule to find $\sin B$; since $\sin B = \sin(180° - B)$, check whether the obtuse angle also fits. $\sin B = \frac{10\sin 40°}{7} = 0.918$. So $B = 66.7°$ or $B = 180° - 66.7° = 113.3°$. Both give a valid angle sum with $A = 40°$ ($40° + 113.3°$), so two triangles exist.

b. Find C from the angle sum, then use the sine rule for c. Triangle 1: $C = 180° - 40° - 66.7° = 73.3°$, $c = \frac{7\sin 73.3°}{\sin 40°}$ = **10.4 cm**. Triangle 2: $C = 180° - 40° - 113.3° = 26.7°$, $c = \frac{7\sin 26.7°}{\sin 40°}$ = **4.89 cm**.

c. The height of the triangle from C is $b\sin A = 10\sin 40° = 6.43$. Two triangles exist when $b\sin A$: 6.43. If $a = 6.43$ there is one right-angled triangle, if a none, and if $a \geq 10$ only one.