Lesson - Straight lines and circles

Year: Year 12

Coordinate geometry joins algebra to pictures. A good diagram helps in nearly every question: sketch the line, the circle and the key points before you calculate.

Starter activity

A radio mast at the point $(3, 4)$ on a map can be heard up to 5 km away. Mark the mast on squared paper with 1 cm for each kilometre and draw the boundary of the listening area with a pair of compasses. Which of these houses can hear the mast: $(6, 8)$, $(7, 7)$, $(2, 0)$? How could you decide without drawing, using only Pythagoras? Write down a rule that works for any house (x, y).

Straight lines

Definition: gradient, distance and midpoint

The **gradient** between (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. The **distance** is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and the **midpoint** is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Rule: equations of a line

$y = mx + c$, or $y - y_1 = m(x - x_1)$, or $ax + by + c = 0$ with integer a, b, c when asked.

Key fact: parallel and perpendicular lines

Parallel lines have the same gradient. Perpendicular lines have gradients with product -1, so if one has gradient m the other has $-\dfrac{1}{m}$.

Worked example 1

Find the perpendicular bisector of $A(1, 2)$, $B(5, 8)$.

Midpoint $(3, 5)$. Gradient of $AB = \dfrac{6}{4} = \dfrac{3}{2}$, so the perpendicular gradient is $-\dfrac{2}{3}$. Then $y - 5 = -\dfrac{2}{3}(x - 3)$, so $3y - 15 = -2x + 6$, giving $2x + 3y - 21 = 0$.

Equation of a circle

Definition: circle

A circle with centre (a, b) and radius r has equation $(x-a)^2+(y-b)^2=r^2$. Expanding gives the general form $x^2+y^2+2fx+2gy+c=0$ with centre $(-f,-g)$ and radius $\sqrt{f^2+g^2-c}$.

Method: centre and radius from a general equation

1. Group the x terms and the y terms.
2. Complete the square for x and for y.
3. Move the constant to the right to get r^2.

Worked example 2

Find the centre and radius of $x^2+y^2-6x+4y-12=0$.

$(x-3)^2-9+(y+2)^2-4-12=0$, so $(x-3)^2+(y+2)^2=25$. Centre $(3,-2)$, radius 5.

Note: testing a point

Substitute the point. If the left-hand side is less than r^2 the point is inside, equal means on the circle, more means outside.

Circle geometry facts

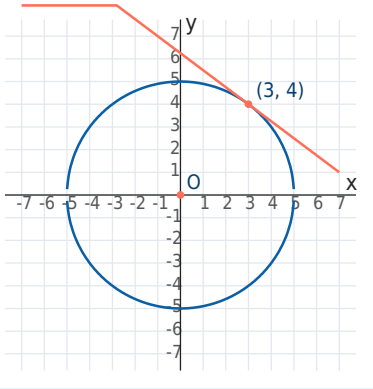
Theorem: circle facts

- The angle in a semicircle is a right angle. If AB is a diameter, the centre is the midpoint of AB.
- The perpendicular from the centre to a chord bisects the chord, so the perpendicular bisector of any chord passes through the centre.
- A tangent is perpendicular to the radius at the point of contact.

Worked example 3 (tangent)

Find the tangent to $x^2+y^2=25$ at $(3,4)$.

The radius from $(0,0)$ has gradient $\frac{4}{3}$, so the tangent has gradient $-\frac{3}{4}$. Then $y-4=-\frac{3}{4}(x-3)$, so $3x+4y=25$.



Line meeting a circle

Method: line and circle

1. Substitute the line equation into the circle equation.
2. Solve the quadratic, or use the discriminant: > 0 the line cuts the circle twice, $= 0$ it is a tangent, it misses the circle.
3. Find the y values from the line equation.

Worked example 4

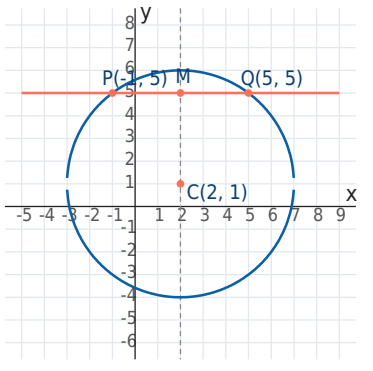
Find where $y = x + 1$ meets $x^2 + y^2 = 13$.

$x^2 + (x+1)^2 = 13$, so $x^2 + x - 6 = 0$ and $x = 2$ or -3. The points are $(2, 3)$ and $(-3, -2)$.

Worked example 5 (chord)

PQ is a chord of $(x-2)^2 + (y-1)^2 = 25$ with midpoint $M(2, 5)$. Show that the perpendicular bisector passes through the centre.

The vertical line $x = 2$ through M is the perpendicular bisector of the horizontal chord, and it goes through the centre $C(2, 1)$.



Circle through three points

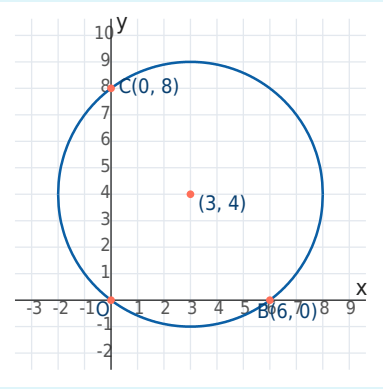
Method: circle through three points

1. Look for a right angle: then the hypotenuse is a diameter.
2. Otherwise find the perpendicular bisectors of two chords and intersect them to get the centre.
3. Find the radius as the distance from the centre to one of the points.

Worked example 6

Find the circle through $O(0,0)$, $B(6,0)$, $C(0,8)$.

The angle at O is 90°, so BC is a diameter. The centre is $(3,4)$ and the radius is 5: $(x-3)^2+(y-4)^2=25$.



Watch out

Centre $(2,-3)$ gives $(x-2)^2+(y+3)^2$: the signs flip. $(x-3)^2+(y-1)^2=16$ has radius 4, not 16. For perpendicular gradients flip *and* change sign. For a tangent always use the radius gradient. Give integer coefficients if the question asks for them.

Worked example 7: Perpendicular bisector of a chord

$A(-1,5)$ and $B(7,1)$ are points on a circle. Find the equation of the perpendicular bisector of AB. The centre of the circle also lies on the line $y=x-1$. Find the centre and the radius.

Midpoint of AB is $(3,3)$ and the gradient of AB is $\dfrac{1-5}{7+1}=-\frac{1}{2}$, so the bisector has gradient 2: $y-3=2(x-3)$, i.e. $y=2x-3$. The centre of any circle through A and B lies on this line. If it also lies on $y=x-1$, then $2x-3=x-1$, so $x=2$ and $y=1$. The centre is $(2,1)$ and the radius is $\sqrt{(2+1)^2+(1-5)^2}=\sqrt{25}=5$. Check with B: $\sqrt{25+0}=5$ ✓.

Worked example 8: Circle with a given diameter

Find the equation of the circle that has the line segment from $(-2, 1)$ to $(6, 5)$ as a diameter, in the form $x^2 + y^2 + px + qy + s = 0$.

The centre is the midpoint $(2, 3)$ and the radius is half the distance: $\frac{1}{2}\sqrt{8^2 + 4^2} = \frac{1}{2}\sqrt{80} = \sqrt{20}$, so $r^2 = 20$. The equation is $(x - 2)^2 + (y - 3)^2 = 20$, which expands to $x^2 - 4x + 4 + y^2 - 6y + 9 = 20$, i.e. $x^2 + y^2 - 4x - 6y - 7 = 0$. Check the end point $(6, 5)$: $36 + 25 - 24 - 30 - 7 = 0$ ✓. A neat shortcut is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, which gives $(x + 2)(x - 6) + (y - 1)(y - 5) = 0$ and the same circle.

Worked example 9: A circle with an unknown constant

The equation $x^2 + y^2 - 4x + 6y + k = 0$ represents a circle. Find the centre and the range of values of k.

Completing the square: $(x - 2)^2 + (y + 3)^2 = 13 - k$. The centre is $(2, -3)$ and $r^2 = 13 - k$. For a genuine circle we need $r^2 > 0$, so k. If $k = 13$ the "circle" is just the single point $(2, -3)$ and if $k > 13$ there are no points at all. For example $k = -12$ gives $r^2 = 25$ and radius 5.

Worked example 10: A tangent through an external point

The line from $T(0, 10)$ touches the circle $x^2 + y^2 = 20$ at a point P in the first quadrant. Find P and the length TP.

Since $OP \perp TP$, triangle OPT has a right angle at P, so $TP^2 = OT^2 - OP^2 = 100 - 20 = 80$ and $TP = \sqrt{80} = 4\sqrt{5} \approx 8.94$. To find $P = (a, b)$: $a^2 + b^2 = 20$ and $TP^2 = a^2 + (b - 10)^2 = 80$. Subtract: $-20b + 100 = 60$, so $b = 2$ and $a^2 = 16$, $a = 4$ (first quadrant). So $P = (4, 2)$. Check the tangent: radius gradient $\frac{1}{2}$, tangent gradient -2, line through $(4, 2)$: $y = -2x + 10$ passes through $T(0, 10)$ ✓.

Worked example 11: Where does a line meet a circle?

Find the points where the line $x + y = 7$ meets the circle $(x - 1)^2 + (y - 2)^2 = 26$.

Substitute $y = 7 - x$: $(x - 1)^2 + (5 - x)^2 = 26$, so $x^2 - 2x + 1 + x^2 - 10x + 25 = 26$, giving $2x^2 - 12x = 0$ and $x(x - 6) = 0$. So $x = 0$ ($y = 7$) or $x = 6$ ($y = 1$). The line meets the circle at $(0, 7)$ and $(6, 1)$. The chord length is $\sqrt{36 + 36} = \sqrt{72} = 6\sqrt{2}$. The midpoint of the chord is $(3, 4)$ and the line from the centre $(1, 2)$ to it has gradient 1, which is perpendicular to the line $x + y = 7$ (gradient -1), as the circle theorems predict.

Worked example 12: Distance from a point to a line using a perpendicular

Find the shortest distance from $(1, 2)$ to the line $y = 2x + 6$.

The line has gradient 2 so the perpendicular has gradient $-\frac{1}{2}$: $y - 2 = -\frac{1}{2}(x - 1)$, i.e.

$y = -\frac{1}{2}x + \frac{5}{2}$. Intersect: $2x + 6 = -\frac{1}{2}x + \frac{5}{2}$, so $\frac{5}{2}x = -\frac{7}{2}$ and $x = -\frac{7}{5}$, $y = \frac{16}{5}$. The distance from $(1, 2)$ to $(-\frac{7}{5}, \frac{16}{5})$ is $\sqrt{(\frac{12}{5})^2 + (\frac{6}{5})^2} = \sqrt{\frac{144+36}{25}} = \sqrt{\frac{180}{25}} = \frac{6\sqrt{5}}{5} \approx 2.68$. This is the radius of the circle centred at $(1, 2)$ that just touches the line.

Worked example 13: Lines that never meet and lines that meet at right angles

The line L_1 has equation $2x - 3y = 6$ and L_2 passes through $(3, 5)$ and is perpendicular to L_1. Find the equation of L_2 and the point where the two lines meet.

Rewrite L_1 as $y = \frac{2}{3}x - 2$ so its gradient is $\frac{2}{3}$. The perpendicular gradient is $-\frac{3}{2}$, so L_2 is $y - 5 = -\frac{3}{2}(x - 3)$, i.e. $3x + 2y = 19$. To find the meeting point solve $2x - 3y = 6$ and $3x + 2y = 19$: multiply the first by 2 and the second by 3 to get $4x - 6y = 12$ and $9x + 6y = 57$. Adding gives $13x = 69$, so $x = \frac{69}{13}$ and $y = \frac{2x-6}{3} = \frac{20}{13}$. Always check that the answer satisfies both original equations.

Method: finding a tangent to a circle

1. Find the centre C of the circle and check that the given point P lies on it.
2. Find the gradient of the radius CP.
3. The tangent is perpendicular to the radius, so its gradient is the negative reciprocal.
4. Use $y - y_1 = m(x - x_1)$ with P as the point and simplify.

Watch out: common errors with lines and circles

- Writing the circle $(x - 3)^2 + (y + 2)^2 = 25$ as having centre $(-3, 2)$. The signs in the brackets are the opposite of the coordinates of the centre: here $(3, -2)$.
- Giving the radius as 25 instead of 5; the right-hand side is r^2.
- Using the gradient of the radius as the gradient of the tangent. The tangent is perpendicular, so use the negative reciprocal.
- Dividing a perpendicular gradient incorrectly: the perpendicular to a gradient $\frac{2}{3}$ line has gradient $-\frac{3}{2}$, not $-\frac{2}{3}$ or $\frac{3}{2}$.
- Forgetting to use the discriminant of the combined equation to decide between tangent, two intersections or none.

Note: the circle theorems you can prove with coordinates

Each theorem in this chapter can be checked algebraically. The angle in a semicircle is 90° because the product of the gradients to the ends of a diameter is -1. A tangent is perpendicular to the radius because a line that meets the circle at exactly one point gives a repeated root in the combined equation. The perpendicular bisector of any chord passes through the centre because every point on it is equidistant from both

ends of the chord. These links between algebra and geometry give you two methods for most questions, so you can always use one to check the other.

Did you know?

Apollonius of Perga, who lived around 200 BC, wrote a book on conic sections that included circles and tangents. It took 1,800 years for René Descartes and Pierre de Fermat to add coordinates, so that the equation $(x-a)^2+(y-b)^2=r^2$ became possible. Today the same equation is used in GPS: your phone works out its position as the point where several circles centred on satellites meet.

Otto's tip

For tangent questions always find the radius gradient first. For chord questions always find the midpoint and go through the centre.