Answers - Straight lines and circles

Year: Year 12

Answer to Question 1

a. Gradient $= \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y_2 - y_1}{x_2 - x_1}$. Gradient $= \dfrac{7-(-1)}{8-2} = \dfrac{8}{6} = \dfrac{4}{3}$. The line rises 4 units for every 3 units across.

b. Use $y - y_1 = m(x - x_1)$ with the point $A(2, -1)$, then clear the fraction and collect the terms on one side. $y + 1 = \frac{4}{3}(x - 2)$. Multiply by 3: $3y + 3 = 4x - 8$, so $4x - 3y - 11 = 0$. Check with B: $32 - 21 - 11 = 0$ ✓.

c. The midpoint is the mean of the x-coordinates and the mean of the y-coordinates. $M = \left(\dfrac{2+8}{2}, \dfrac{-1+7}{2}\right) = (5, 3)$.

d. Use Pythagoras with $\Delta x = 6$ and $\Delta y = 8$. $AB = \sqrt{6^2 + 8^2} = \sqrt{100} = \mathbf{10}$. So AB is 10 units long.

Answer to Question 2

a. Rearrange into the form $y = mx + c$. $4y = -3x + 12$, so $y = -\frac{3}{4}x + 3$ and the gradient is $-\frac{3}{4}$.

b. Parallel lines have the same gradient. $y + 2 = -\frac{3}{4}(x - 4)$, so $4y + 8 = -3x + 12$ and $3x + 4y = 4$.

c. Perpendicular gradients multiply to -1, so the new gradient is $-1 \div \left(-\frac{3}{4}\right) = \frac{4}{3}$. The perpendicular gradient is $\frac{4}{3}$: $y = \frac{4}{3}x + 5$, i.e. $4x - 3y + 15 = 0$.

d. The foot of the perpendicular is where the perpendicular line from part (c) meets L; solve the two equations simultaneously. Solve $3x + 4y = 12$ and $4x - 3y = -15$. Multiply: $9x + 12y = 36$ and $16x - 12y = -60$. Add: $25x = -24$, $x = -\frac{24}{25}$, $y = \frac{93}{25}$. Foot $\left(-\frac{24}{25}, \frac{93}{25}\right)$. Distance $= \sqrt{\frac{576}{625} + \left(5 - \frac{93}{25}\right)^2} = \sqrt{\frac{576+1024}{625}} = \sqrt{\frac{1600}{625}} = \frac{8}{5} = 1.6$. So the shortest distance from $(0, 5)$ to L is 1.6 units.

Answer to Question 3

a. A circle with centre (a, b) and radius r has equation $(x - a)^2 + (y - b)^2 = r^2$. $(x - 3)^2 + (y + 2)^2 = 25$. Note the signs: the centre $(3, -2)$ gives $(x - 3)$ and $(y + 2)$.

b. Expand both brackets and collect all terms on one side. $x^2 - 6x + 9 + y^2 + 4y + 4 = 25$, so $x^2 + y^2 - 6x + 4y - 12 = 0$. This is the required form.

c. Substitute each point into the left-hand side of the equation and compare with $r^2 = 25$. $(6, 2)$: $3^2 + 4^2 = 25$ ✓. $(0, 2)$: $(-3)^2 + 4^2 = 25$ ✓. Both points are on the circle.
d. Compare the squared distance from the centre with r^2: less means inside, more means outside. $(7, 1)$: $(7-3)^2 + (1+2)^2 = 16 + 9 = 25$. This equals r^2, so the point lies **on** the circle.

Answer to Question 4

a. Complete the square in x and in y. Complete the square: $(x-4)^2 - 16 + (y+3)^2 - 9 - 11 = 0$, so $(x-4)^2 + (y+3)^2 = 36$. Centre $(4, -3)$, radius **6**.
b. Use the same method. $(x+1)^2 - 1 + (y-2)^2 - 4 - 4 = 0$, so $(x+1)^2 + (y-2)^2 = 9$. Centre $(-1, 2)$, radius **3**.
c. Two circles meet at two points when the distance between the centres lies between the difference and the sum of the radii. Distance $= \sqrt{5^2 + 5^2} = 5\sqrt{2} \approx 7.07$. The sum of radii is 9 and the difference is 3. Since 3 the circles **intersect** at two points.

Answer to Question 5

a. The centre of the circle is the midpoint of the diameter, and the radius is half the length of the diameter. Centre = midpoint $= (4, 6)$. $AB = \sqrt{36 + 64} = 10$, so the radius is **5**.
b. $(x-4)^2 + (y-6)^2 = 25$.
c. Substitute C into the equation. $(8-4)^2 + (9-6)^2 = 16 + 9 = 25$ ✓. So C lies on the circle.
d. Gradient of $AC = \dfrac{9-2}{8-1} = 1$ and gradient of $BC = \dfrac{9-10}{8-7} = -1$. The product is -1 so $AC \perp BC$. This illustrates the **angle in a semicircle is a right angle** theorem.

Answer to Question 6

a. Substitute the point into the equation. $(4-1)^2 + (2+2)^2 = 9 + 16 = 25$ ✓.
b. The centre is $(1, -2)$. Centre $(1, -2)$: gradient $= \dfrac{2-(-2)}{4-1} = \dfrac{4}{3}$.
c. A tangent is perpendicular to the radius at the point of contact. The tangent is perpendicular to the radius: gradient $-\frac{3}{4}$. $y - 2 = -\frac{3}{4}(x-4)$, $4y - 8 = -3x + 12$, so $3x + 4y = 20$.
d. Find both intercepts of the tangent. When $y = 0$: $x = \frac{20}{3}$, so $T(\frac{20}{3}, 0)$. When $x = 0$: $y = 5$. Area $= \frac{1}{2} \times \frac{20}{3} \times 5 = \frac{50}{3} \approx 16.7$ square units.

Answer to Question 7

a. The mast is the centre and the range 5 km is the radius. $(x-3)^2+(y-4)^2=25$.

b. Substitute the equation of the road into the equation of the circle. Substitute $y=x-2$: $(x-3)^2+(x-6)^2=25$, so $2x^2-18x+45=25$, giving $2x^2-18x+20=0$, i.e. $x^2-9x+10=0$. This is the required equation.

c. Use the quadratic formula; the road has gradient 1, so the chord length is $\sqrt{2}$ times the difference in x. $x=\dfrac{9\pm\sqrt{81-40}}{2}=\dfrac{9\pm\sqrt{41}}{2}$. The x-values differ by $\sqrt{41}$ and $y=x-2$ so the y-values also differ by $\sqrt{41}$. Length $=\sqrt{41+41}=\sqrt{82}=$ **9.06 km**. About 9.06 km of road has coverage.

d. The shortest distance is along the perpendicular from M to the road. The perpendicular from M has gradient -1: $y=-x+7$. It meets the road where $x-2=-x+7$, $x=4.5$, $y=2.5$. Distance $=\sqrt{1.5^2+1.5^2}=\sqrt{4.5}=$ **2.12 km**. Since 2.12, this confirms the road passes through the coverage area.

Answer to Question 8

a. Compare with $(x-a)^2+(y-b)^2=r^2$. Centre $(6,5)$, radius 3 m.

b. From the path, $y=\dfrac{53-3x}{4}$. Then $y-5=\dfrac{33-3x}{4}=\dfrac{3(11-x)}{4}$. Substituting: $(x-6)^2+\dfrac{9}{16}(x-11)^2=9$. Multiply by 16: $16x^2-192x+576+9x^2-198x+1089=144$, so $25x^2-390x+1521=0$. The discriminant is $390^2-4\times25\times1521=152100-152100=0$, so there is exactly one point of contact: the path is a **tangent**.

c. $x=\dfrac{390}{50}=7.8$ and $y=\dfrac{53-23.4}{4}=7.4$. Contact point $(7.8,\ 7.4)$. (Check: $1.8^2+2.4^2=9$ ✓.)

d. The other tangent is on the opposite side of the centre: the centre has $3x+4y=18+20=38$, and $53-38=15$, so the other is $38-15=23$: $3x+4y=23$.

Answer to Question 9

a. The centre is at $(0,12)$ and the radius is 10. $x^2+(y-12)^2=100$. Every cabin is 10 m from the centre of the wheel.

b. Substitute $x=6$ and solve for y. $x=6$: $(y-12)^2=64$, so $y-12=\pm8$ and $y=20$ or $y=4$. Heights **20 m** and **4 m**. One position is above the centre and one below.

c. The lowest point is directly below the centre, one radius down. Lowest point: $12-10=$ **2 m** above the ground. The lowest cabin is 2 m above the ground.

d. Substitute $x = 5$ and take the positive square root, because the cabin is in the upper half. $(y - 12)^2 = 75$, $y = 12 + \sqrt{75} = 12 + 8.66 =$ **20.7 m**. The cabin is about 20.7 m above the ground.

Answer to Question 10

a. The perpendicular bisector of a chord passes through the centre of the circle. AB is horizontal with midpoint $(4, 0)$, so its perpendicular bisector is $x = 4$. AC is vertical with midpoint $(0, 3)$, so its perpendicular bisector is $y = 3$.
b. They meet at the centre $(4, 3)$. Radius $= \sqrt{4^2 + 3^2} =$ **5 m**.
c. Expand the equation in centre-radius form. $(x - 4)^2 + (y - 3)^2 = 25 \Rightarrow$ $x^2 + y^2 - 8x - 6y = 0$. (Check B: $64 - 64 = 0$ ✓; C: $36 - 36 = 0$ ✓.)
d. $BC = \sqrt{64 + 36} = 10 = 2 \times 5$, so BC is a **diameter**. The angle at A is 90° (angle in a semicircle).

Answer to Question 11

a. Find the gradient of the road, then use $y - y_1 = m(x - x_1)$; the fence has the same gradient. Road gradient $= \dfrac{5 - 2}{6 - 0} = \frac{1}{2}$: $y = \frac{1}{2}x + 2$. Fence: $y - 6 = \frac{1}{2}(x - 2)$, so $y = \frac{1}{2}x + 5$. The road is $y = \frac{1}{2}x + 2$ and the fence is $y = \frac{1}{2}x + 5$.
b. A track at right angles to the road has gradient $-1 \div \frac{1}{2} = -2$. Perpendicular gradient -2: $y - 6 = -2(x - 2)$, so $y = -2x + 10$. The track is $y = -2x + 10$.
c. Solve the equations of the track and the road simultaneously. $\frac{1}{2}x + 2 = -2x + 10$ gives $2.5x = 8$, $x = 3.2$, $y = 3.6$. Point $(3.2,\ 3.6)$. The track meets the road at $(3.2,\ 3.6)$.
d. The shortest distance between the parallel lines is the length of the perpendicular track. Distance $= \sqrt{(3.2 - 2)^2 + (3.6 - 6)^2} = \sqrt{1.44 + 5.76} = \sqrt{7.2} =$ **2.68 m**. The fence is about 2.68 m from the road.

Answer to Question 12

a. Two circles overlap when the distance between centres is between the difference and the sum of the radii. Centres $(2, 3)$ and $(8, 3)$: distance $= 6$. Radii 4 and 3: $4 - 3 = 1$, so the circles overlap.
b. Subtracting the equations removes the x^2 and y^2 terms. Subtract the equations: $(x - 2)^2 - (x - 8)^2 = 16 - 9 = 7$. $(x^2 - 4x + 4) - (x^2 - 16x + 64) = 12x - 60 = 7$, so $x = \dfrac{67}{12} = 5.583$.
c. Substitute $x = \frac{67}{12}$ into the first circle. $(x - 2)^2 = \left(\frac{43}{12}\right)^2 = 12.84$, so $(y - 3)^2 = 16 - 12.84 = 3.16$, $y - 3 = \pm 1.78$. So $y =$ **4.78** or **1.22**.

d. The two intersection points have the same x-coordinate, so the chord is vertical. Chord length $= 2 \times 1.78 =$ **3.56 m**. The common chord is about 3.56 m long.

Answer to Question 13

a. Substitute P into the equation of the circle. $(5-2)^2 + (3-1)^2 = 9 + 4 = 13$ ✓. So P lies on the boundary.
b. The tangent is perpendicular to the radius from the centre $(2, 1)$ to P. Radius gradient $= \dfrac{3-1}{5-2} = \frac{2}{3}$. Tangent gradient $= -\frac{3}{2}$. $y - 3 = -\frac{3}{2}(x-5)$, so $2y - 6 = -3x + 15$, giving $3x + 2y = 21$. The tangent is $3x + 2y = 21$.
c. Set $y = 0$ and then $x = 0$ in the equation of the tangent. $y = 0$: $x = 7$. $x = 0$: $y = 10.5$. The ends are $(7, 0)$ and $(0, 10.5)$.
d. Use Pythagoras with the two intercepts. Length $= \sqrt{7^2 + 10.5^2} = \sqrt{159.25} =$ **12.62 km**. The track is about 12.62 km long.

Answer to Question 14

a. Use the midpoint formula and the gradient formula. Midpoint $(5, 4)$. Gradient $= \dfrac{6-2}{9-1} = \frac{1}{2}$. The midpoint is $(5, 4)$ and the gradient is $\frac{1}{2}$.
b. Every point equidistant from A and B lies on the perpendicular bisector, which passes through the midpoint with gradient -2. Perpendicular gradient -2: $y - 4 = -2(x-5)$, so $y = -2x + 14$. The perpendicular bisector is $y = -2x + 14$.
c. The bus stop is on both the perpendicular bisector and the road, so solve simultaneously. $x + 2 = -2x + 14$ gives $3x = 12$, $x = 4$, $y = 6$. The bus stop is $(4, 6)$.
d. Use Pythagoras for each distance. To A: $\sqrt{3^2 + 4^2} = 5$. To B: $\sqrt{5^2 + 0^2} = 5$. ✓ The bus stop is 5 km from each village.

Answer to Question 15

a. The perpendicular bisector of a chord passes through the centre. PQ is horizontal with midpoint $(4, 1)$, so the bisector is $x = 4$.
b. PR has gradient $\dfrac{5-1}{8-2} = \frac{2}{3}$ and midpoint $(5, 3)$. The bisector has gradient $-\frac{3}{2}$: $y - 3 = -\frac{3}{2}(x-5)$.
c. The centre lies on both bisectors. At $x = 4$: $y = 3 + \frac{3}{2} = 4.5$. Centre $(4, 4.5)$. Radius$^2 = (2-4)^2 + (1-4.5)^2 = 4 + 12.25 = 16.25$, so the radius is $\sqrt{16.25} = \dfrac{\sqrt{65}}{2} \approx 4.03$ **m**.
d. Substitute R into $(x-4)^2 + (y-4.5)^2 = 16.25$. R:

$(8-4)^2+(5-4.5)^2=16+0.25=16.25$ ✓. Area $=\pi r^2=16.25\pi=$ **51.1 m**^{2}.

Answer to Question 16

a. Substitute the line into the circle and expand $(2x+k)^2=4x^2+4kx+k^2$. Substitute: $x^2+(2x+k)^2=20\Rightarrow 5x^2+4kx+k^2-20=0$.
b. A tangent gives equal roots, so the discriminant is zero. Discriminant $=16k^2-20(k^2-20)=400-4k^2=0$, so $k^2=100$ and $k=\pm 10$. (For $k=10$ the contact is at $(-4,2)$.)
c. Two points need $400-4k^2>0$: -10.
d. $5x^2+20x+5=0\Rightarrow x^2+4x+1=0\Rightarrow x=-2\pm\sqrt{3}$. The points are $(-2+\sqrt{3},\,1+2\sqrt{3})$ and $(-2-\sqrt{3},\,1-2\sqrt{3})$. The x-difference is $2\sqrt{3}$ and the y-difference is $4\sqrt{3}$, so the chord is $\sqrt{12+48}=\sqrt{60}=2\sqrt{15}\approx 7.75$.

Answer to Question 17

a. Complete the square in x and in y. $(x-3)^2+(y-2)^2=25$. Centre $(3,2)$, radius **5**.
b. $36+36-36-24-12=0$ ✓. Radius gradient $\dfrac{6-2}{6-3}=\frac{4}{3}$; tangent gradient $-\frac{3}{4}$: $y-6=-\frac{3}{4}(x-6)$, so $3x+4y=42$.
c. $y=0$: $x=14$. $T=(14,0)$.
d. $TP=\sqrt{(14-6)^2+(0-6)^2}=\sqrt{64+36}=10$ ✓. The distance from T to the centre is $\sqrt{11^2+2^2}=\sqrt{125}$. By Pythagoras the tangent length is $\sqrt{125-25}=10$ for either tangent from T, so the other tangent is also 10.

Answer to Question 18

a. Centres $(0,0)$ and $(6,0)$, each with radius 5.
b. Subtract: $x^2-(x-6)^2=0\Rightarrow 12x-36=0$, so $x=3$. Then $y^2=25-9=16$, $y=\pm 4$. Points $(3,4)$ and $(3,-4)$.
c. The chord from $(3,-4)$ to $(3,4)$ has length **8 km**. It is vertical, and the line of centres is horizontal, so they are perpendicular.
d. The midpoint of the centres is $(3,0)$, which lies on the line $x=3$. ✓

Answer to Question 19

a. Substitute each point into x^2+y^2. $3^2+4^2=25$, $(-4)^2+3^2=25$, $0+(-5)^2=25$ ✓.
b. P: $AP=\frac{4}{8}=\frac{1}{2}$, $BP=\frac{4}{-2}=-2$; product -1. Q: $AQ=\frac{3}{1}=3$, $BQ=\frac{3}{-9}=-\frac{1}{3}$;

product -1. R: $AR = \frac{-5}{5} = -1$, $BR = \frac{-5}{-5} = 1$; product -1.

c. Gradient $TA = \dfrac{y}{x+5}$ and gradient $TB = \dfrac{y}{x-5}$. Their product is $\dfrac{y^2}{x^2-25}$. Since $x^2 + y^2 = 25$, $x^2 - 25 = -y^2$, so the product is $\dfrac{y^2}{-y^2} = -1$ (for $y \neq 0$). Hence $TA \perp TB$ for every point on the circle: the angle in a semicircle is a right angle.

Answer to Question 20

a. The circle touches the x-axis, so the distance from the centre to the axis equals the radius: the centre is at height a, so $r = a$. Equation: $(x-a)^2 + (y-a)^2 = a^2$.

b. Substitute $(8, 4)$:
$(8-a)^2 + (4-a)^2 = a^2 \Rightarrow 64 - 16a + a^2 + 16 - 8a + a^2 = a^2 \Rightarrow a^2 - 24a + 80 = 0$
. $(a-4)(a-20) = 0$, so $a = 4$ or 20. Circles: $(x-4)^2 + (y-4)^2 = 16$ and $(x-20)^2 + (y-20)^2 = 400$.

c. Centres $(4, 4)$ and $(20, 20)$: distance $= 16\sqrt{2} \approx 22.6$. Radii 4 and 20 have sum 24 and difference 16. Since 16 the circles intersect at two points. Both circles are symmetrical about the line $y = x$ (their centres lie on it), so since $(8, 4)$ is an intersection point, its reflection $(4, 8)$ is the other. Check: $(4-4)^2 + (8-4)^2 = 16$ ✓ and $(4-20)^2 + (8-20)^2 = 256 + 144 = 400$ ✓.