Lesson - Simultaneous equations, inequalities and polynomials

Year: Year 12

These three topics are the heart of AS algebra. They share one idea: sketch or factorise, then read the answer from the structure.

Starter activity

A café sells 2 teas and 3 muffins for £9.60 and 3 teas and 1 muffin for £6.30. Without writing any algebra, try to guess the price of a tea and a muffin. Then write the information as two equations using t and m. Now think about how you can remove one letter at a time. Which of your guesses was closest, and how could you tell without trying every price?

Simultaneous equations

Definition: simultaneous equations

Two or more equations that must be true at the same time. Their solutions are the points where the graphs meet.

Method: one linear and one quadratic equation

1. Rearrange the linear equation to make one variable the subject.
2. Substitute into the quadratic and solve.
3. Find the other variable from the *linear* equation (this avoids false pairs).

Worked example 1

Solve $3x + 2y = 16$ and $x - y = 2$.

Write $x = y + 2$, so $3(y + 2) + 2y = 16$, $5y = 10$, $y = 2$ and $x = 4$.

Worked example 2

Solve $y = x + 1$ and $x^2 + y^2 = 25$.

Substitute: $x^2 + (x + 1)^2 = 25$, so $2x^2 + 2x - 24 = 0$, $x^2 + x - 12 = 0$, $(x + 4)(x - 3) = 0$. Solutions: $x = 3, y = 4$ and $x = -4, y = -3$.

Key fact: intersections and the discriminant

After substituting you get a quadratic. If its discriminant is > 0 there are two

intersections, if $= 0$ the line is a tangent, if there is no intersection.

Worked example 3

Find k if $y = 3x + k$ is a tangent to $y = x^2 + 2x + 5$.

Setting equal: $x^2 - x + (5 - k) = 0$. Tangent means $b^2 - 4ac = 0$: $1 - 4(5 - k) = 0$, so $k = \frac{19}{4}$.

Watch out

Do not stop after finding x: give the matching y values too.

Inequalities

Rule: linear inequalities

Solve like an equation, but **reverse the inequality sign when you multiply or divide by a negative number**. For $5 - 2x \geq 13$: $-2x \geq 8$, so $x \leq -4$.

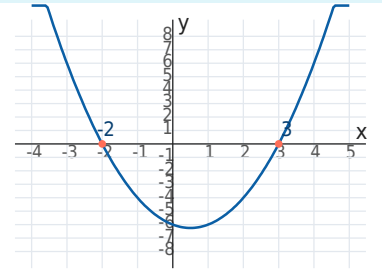
Method: quadratic inequalities

1. Solve the equation to find the roots.
2. Sketch the $\cup$ or $\cap$ shape through the roots.
3. Read off where the graph is above (> 0) or below () the x-axis.

Worked example 4

Solve $x^2 - x - 6$.

$(x - 3)(x + 2)$, roots -2 and 3. The $\cup$-shaped curve is below the axis between the roots, so -2.



Worked example 5

Solve $x^2 + 3x - 10 \geq 0$.

The roots are -5 and 2 and the curve is above the axis outside them: $x \leq -5$ or $x \geq 2$.

In set notation $\{x : x \leq -5\} \cup \{x : x \geq 2\}$.

Watch out

Never write $-5 \geq x \geq 2$, which is impossible. Never divide by an expression containing x: it may be negative or zero. When two conditions must *both* hold, take the overlap.

Worked example 6

Find the values of x with $x^2 - 5x + 4$ *and* $3x - 2 > 1$.

First: $(x - 1)(x - 4)$, so 1. Second: $3x > 3$, so $x > 1$. Both must hold, so the answer is 1.

Polynomials

Definition: polynomial

A sum of terms ax^n with whole-number powers. The highest power is the **degree**; degree 3 is a *cubic*.

Theorem: remainder and factor theorems

When $f(x)$ is divided by $(x - a)$ the remainder is $f(a)$. Also $(x - a)$ is a factor of $f(x)$ if and only if $f(a) = 0$. For a factor $(px - a)$ test $f(a/p)$.

Method: factorising a cubic

1. Try $x = \pm 1, \pm 2, \pm 3, \ldots$ (a root divides the constant term) until $f(a) = 0$.
2. Write $f(x) = (x - a)(x^2 + px + q)$ and match coefficients, or divide.
3. Factorise the quadratic.

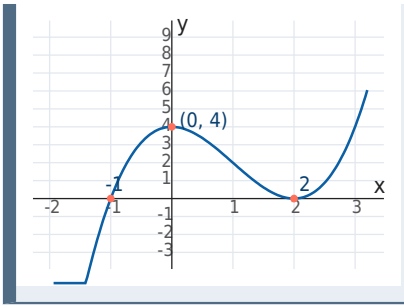
Worked example 7

Factorise $f(x) = x^3 - 3x^2 - 4x + 12$.

$f(2) = 8 - 12 - 8 + 12 = 0$, so $(x - 2)$ is a factor. Write $x^3 - 3x^2 - 4x + 12 = (x - 2)(x^2 + px + q)$: $p - 2 = -3$ gives $p = -1$, and $-2q = 12$ gives $q = -6$. Check the x term: $q - 2p = -4$. So $f(x) = (x - 2)(x - 3)(x + 2)$.

Note: sketching a cubic

Find the roots, the y-intercept and the end behaviour. A repeated factor means the graph *touches* the axis. The graph of $y = (x + 1)(x - 2)^2$ crosses at $x = -1$, touches at $x = 2$ and meets the y-axis at 4.



Watch out

For the factor $(x - a)$ test $f(a)$, not $f(-a)$. In an exam write "$f(2) = 0$, so $(x - 2)$ is a factor": the words carry a mark.

Worked example 8: Simultaneous equations by elimination

Solve $3x + 2y = 16$ and $5x - 4y = 6$.

Multiply the first equation by 2 so that the y coefficients match in size: $6x + 4y = 32$. Add this to the second equation: $6x + 4y + 5x - 4y = 32 + 6$, so $11x = 38$ and $x = \frac{38}{11}$. Substitute back: $3 \times \frac{38}{11} + 2y = 16$ gives $2y = \frac{176 - 114}{11} = \frac{62}{11}$, so $y = \frac{31}{11}$. Always check in the other equation: $5 \times \frac{38}{11} - 4 \times \frac{31}{11} = \frac{190 - 124}{11} = 6$ ✓.

Worked example 9: A line meeting a curve at two points

Find where the line $y = x + 1$ meets the curve $y = x^2 - 3x + 5$ and the length of the chord between the points.

Equate: $x^2 - 3x + 5 = x + 1$, so $x^2 - 4x + 4 = 0$, $(x - 2)^2 = 0$. There is one repeated solution $x = 2$, $y = 3$: the line is a **tangent** at $(2, 3)$ and there is no chord. Change the line to $y = x - 1$: $x^2 - 4x + 6 = 0$ has discriminant $16 - 24$, so there are no intersections. Try $y = x + 3$: $x^2 - 4x + 2 = 0$, $x = 2 \pm \sqrt{2}$, giving two points whose distance is $\sqrt{2 \times (2\sqrt{2})^2} = 4$.

Worked example 10: Algebraic long division by a linear factor

Divide $2x^3 - 5x^2 - 4x + 3$ by $(x - 3)$ and use the result to solve $2x^3 - 5x^2 - 4x + 3 = 0$.

$2x^3 \div x = 2x^2$; subtract $2x^3 - 6x^2$ to leave $x^2 - 4x$. $x^2 \div x = x$; subtract $x^2 - 3x$ to leave $-x + 3$. $-x \div x = -1$; subtract $-x + 3$ to leave 0. So the quotient is $2x^2 + x - 1$ with no remainder, and $2x^3 - 5x^2 - 4x + 3 = (x - 3)(2x^2 + x - 1) = (x - 3)(2x - 1)(x + 1)$. The roots are $x = 3$, $x = \frac{1}{2}$ and $x = -1$.

Worked example 11: Dividing by a non-monic linear factor

Find the remainder when $6x^3 + x^2 - 5x + 4$ is divided by $(2x - 1)$.

By the remainder theorem, divide by writing $2x - 1 = 2(x - \frac{1}{2})$ and evaluate at $x = \frac{1}{2}$. $p(\frac{1}{2}) = 6 \times \frac{1}{8} + \frac{1}{4} - \frac{5}{2} + 4 = \frac{3}{4} + \frac{1}{4} - \frac{5}{2} + 4 = 2.5$. So the remainder is $\frac{5}{2}$. Check by division: $6x^3 + x^2 - 5x + 4 = (2x - 1)(3x^2 + 2x - \frac{3}{2}) + \frac{5}{2}$. The key idea is that the remainder on division by $(ax - b)$ is $p\left(\frac{b}{a}\right)$.

Worked example 12: A cubic inequality

Solve $(x + 2)(x - 1)(x - 4) \geq 0$.

The critical values are -2, 1 and 4. The cubic has a positive x^3 coefficient, so it is positive for $x > 4$, and the sign alternates as you move left past each root: negative for 1, positive for -2, negative for x. The inequality allows equality, so the roots are included: $-2 \leq x \leq 1$ or $x \geq 4$. Test $x = 0$: $2 \times (-1) \times (-4) = 8 > 0$ ✓, test $x = 2$: $4 \times 1 \times (-2) = -8$ ✓ (so excluded).

Worked example 13: Describing a region

Describe the region defined by $y > x^2 - 4$ and $y \leq x + 2$.

Draw the parabola $y = x^2 - 4$ as a dashed curve (strict inequality, boundary not included) and the line $y = x + 2$ as a solid line (boundary included). The line meets the parabola where $x^2 - 4 = x + 2$, i.e. $x^2 - x - 6 = 0$, $(x - 3)(x + 2) = 0$, so $x = 3$ and $x = -2$, giving points $(3, 5)$ and $(-2, 0)$. The region lies above the parabola and on or below the line: it is the finite lens-shaped area between $x = -2$ and $x = 3$. Test the origin: $0 > -4$ ✓ and $0 \leq 2$ ✓, so the origin is inside the region.

Worked example 14: Finding the equation of a cubic from its roots

A cubic curve crosses the x-axis at $x = -1$, $x = 2$ and $x = 5$ and passes through the point $(0, 20)$. Find its equation and the height of the curve at $x = 3$.

Write the cubic in factored form $y = k(x + 1)(x - 2)(x - 5)$. At $x = 0$: $k(1)(-2)(-5) = 10k = 20$, so $k = 2$. The equation is $y = 2(x + 1)(x - 2)(x - 5)$, which expands to $y = 2x^3 - 12x^2 + 6x + 20$. At $x = 3$: $y = 2 \times 4 \times 1 \times (-2) = -16$, so the curve is 16 units below the axis there, which agrees with the sign pattern: the curve is negative between the roots 2 and 5.

Worked example 15: A repeated factor and a tangent to the axis

Given that $(x - 3)^2$ is a factor of $x^3 - 5x^2 + 3x + 9$, find the remaining factor and sketch the curve.

Since $(x - 3)^2 = x^2 - 6x + 9$ is a factor, divide: $x^3 - 5x^2 + 3x + 9 = (x^2 - 6x + 9)(x + 1)$. Check by expanding: $x^3 - 6x^2 + 9x + x^2 - 6x + 9 = x^3 - 5x^2 + 3x + 9$ ✓. So $y = (x - 3)^2(x + 1)$. The curve

crosses the axis at $x = -1$ and **touches** it at $x = 3$ (repeated root), with y-intercept 9. For $x > -1$ the curve is never negative because $(x-3)^2 \geq 0$, so the curve rises from $(-1, 0)$, turns at a maximum, comes down to touch the axis at $(3, 0)$ and then rises again.

Worked example 16: An inequality in a modelling context

A rectangle has width x cm and length $(2x + 5)$ cm. The area must be greater than 75 cm^2 but the perimeter must not exceed 60 cm. Find the possible values of x.

Area: $x(2x + 5) > 75$ gives $2x^2 + 5x - 75 > 0$. Use the formula: $x = \dfrac{-5 \pm \sqrt{25 + 600}}{4} = \dfrac{-5 \pm 25}{4}$, so $x = 5$ or $x = -7.5$. The U-shape is positive outside the roots; since $x > 0$ we need $x > 5$. Perimeter: $2(x + 2x + 5) = 6x + 10 \leq 60$ gives $x \leq \frac{25}{3} \approx 8.33$. Both conditions: 5. For instance $x = 6$ gives a rectangle 6 cm by 17 cm with area 102 cm^2 and perimeter 46 cm.

Note: the sign diagram

A **sign diagram** is a quick way to solve polynomial inequalities without drawing a full graph. Mark the roots in order on a number line. For a polynomial whose highest-power term has a positive coefficient, the sign on the far right is positive; as you cross each simple root the sign changes, but when you cross a repeated (even) root the sign does not change. Read off the intervals you need. This method is reliable for quadratics, cubics and quartics, and the same idea is used later with the signs of derivatives to classify stationary points.

Method: solving a polynomial equation by the factor theorem

1. Try small integer values ($\pm 1, \pm 2, \pm 3$ and factors of the constant term) until $f(a) = 0$.
2. Then $(x - a)$ is a factor. Divide $f(x)$ by $(x - a)$ using long division or by inspection.
3. Factorise the remaining quadratic.
4. Write all the roots and, if asked, sketch the curve or solve an inequality using the sign pattern.

Watch out: common errors

- Not reversing the inequality sign when multiplying or dividing by a negative number.
- Solving $x^2 > 9$ as $x > 3$ only. The full answer is x or $x > 3$.
- Using the factor theorem with the wrong sign: the factor $(x + 2)$ means $f(-2) = 0$, not $f(2) = 0$.

- Giving only the x-values of simultaneous equations. Always find the matching y-values and pair them up.
- Forgetting that a repeated root means the curve touches the axis without crossing.

Did you know?

The word *algebra* comes from the Arabic *al-jabr*, from the title of a book by the Persian mathematician Muhammad ibn Musa al-Khwarizmi written around 820. The method of eliminating one unknown from a pair of simultaneous equations is found in Chinese texts more than 2,000 years old, and Gauss rediscovered the idea in 1809 for the equations used to track the asteroid Ceres.

Otto's tip

When stuck on a cubic, try $x = \pm 1, \pm 2, \pm 3$ in the factor theorem. If the constant term is 6, a root must divide 6.