Answers - Simultaneous equations, inequalities and polynomials

Year: Year 12

Answer to Question 1

a. Both equations are linear, so use elimination: make the coefficients of x equal. Multiply the second equation by 2: $4x + 10y = 2$. Subtract from the first: $-13y = 13$, so $y = -1$. Then $2x - 5 = 1$ gives $x = 3$. Solution: $x = 3$, $y = -1$. Check: $12 + 3 = 15$ ✓ and $6 - 5 = 1$ ✓.

b. One equation is linear and one is quadratic, so substitute the linear one into the quadratic. Substitute: $x^2 + (2x - 1)^2 = 10$, so $5x^2 - 4x - 9 = 0$, $(5x - 9)(x + 1) = 0$, giving $x = \frac{9}{5}$ or $x = -1$. Then $y = 2x - 1$: $(\frac{9}{5}, \frac{13}{5})$ and $(-1, -3)$. Check: $1 + 9 = 10$ ✓.

c. Make y the subject of the linear equation and substitute into $xy = 12$. From the second equation $y = 7 - x$. Then $x(7 - x) = 12$, so $x^2 - 7x + 12 = 0$, $(x - 3)(x - 4) = 0$. $(3, 4)$ and $(4, 3)$.

Answer to Question 2

a. Collect the terms, then divide by the coefficient of x. $-3x \geq 15$, and dividing by -3 reverses the sign: $x \leq -5$.

b. Expand the bracket and collect the x terms on one side. $2x + 8$, so 9, i.e. $x > 9$.

c. Factorise to find the critical values, then use a sketch of the parabola. $x^2 - x - 12 = (x - 4)(x + 3) > 0$. The graph is U-shaped and positive outside the roots: x **or** $x > 4$.

d. Do not divide by x, because x could be negative or zero; bring every term to one side and factorise instead. $3x^2 - 12x \leq 0$, $3x(x - 4) \leq 0$. U-shape, negative between the roots: $0 \leq x \leq 4$.

Answer to Question 3

a. By the factor theorem, $(x - a)$ is a factor if $f(a) = 0$; try small divisors of the constant term 6. $f(1) = 1 - 7 + 6 = 0$, so $(x - 1)$ is a factor. Divide: $x^3 - 7x + 6 = (x - 1)(x^2 + x - 6)$, and $x^2 + x - 6 = (x + 3)(x - 2)$. So $f(x) = (x - 1)(x - 2)(x + 3)$. Check: $f(2) = 8 - 14 + 6 = 0$ ✓.

b. Try divisors of 6 again: $g(1) = -4 \neq 0$, but $g(2) = 0$. $g(2) = 16 + 4 - 26 + 6 = 0$, so $(x - 2)$ is a factor. Divide: $g(x) = (x - 2)(2x^2 + 5x - 3)$ and $2x^2 + 5x - 3 = (2x - 1)(x + 3)$. So $g(x) = (x - 2)(2x - 1)(x + 3)$. Check: $g(\frac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{13}{2} + 6 = 0$ ✓.

Answer to Question 4

a. Use long division, dividing the leading term each time.
$x^3 + 2x^2 - 5x + 1 = (x - 2)(x^2 + 4x + 3) + 7$. Quotient $x^2 + 4x + 3$, remainder **7**.
(Working: $x^3 \div x = x^2$, subtract $x^3 - 2x^2$ to leave $4x^2 - 5x$; $4x^2 \div x = 4x$, subtract $4x^2 - 8x$ to leave $3x + 1$; $3x \div x = 3$, subtract $3x - 6$ to leave 7.)

b. By the remainder theorem, the remainder on dividing by $(x - a)$ is $f(a)$.
$f(2) = 8 + 8 - 10 + 1 = 7$. ✓ This agrees with the long division.

c. Dividing by $(x + 1)$ means $a = -1$, so evaluate $f(-1)$.
$f(-1) = -1 + 2 + 5 + 1 = 7$. The remainder is **7** (a coincidence in this case).

Answer to Question 5

a. Use the factor theorem for $(x - 2)$ and the remainder theorem for $(x + 1)$.
$p(2) = 0$: $8 + 4a + 2b - 6 = 0$, so $2a + b = -1$. $p(-1) = -12$:
$-1 + a - b - 6 = -12$, so $a - b = -5$.

b. Add the two equations to eliminate b. Add: $3a = -6$, so $a = -2$ and $b = 3$. Check:
$2(-2) + 3 = -1$ ✓ and $-2 - 3 = -5$ ✓.

c. Factorise $p(x)$ by grouping.
$p(x) = x^3 - 2x^2 + 3x - 6 = x^2(x - 2) + 3(x - 2) = (x - 2)(x^2 + 3)$. The quadratic $x^2 + 3$ has discriminant $0 - 12$, so no real roots. The only real root is $x = 2$. So the graph of $y = p(x)$ crosses the x-axis only once, at $x = 2$.

Answer to Question 6

a. Solve each inequality separately, then find the values that satisfy both, for example on a number line. $2x > -6$ gives $x > -3$. $x^2 \leq 16$ gives $-4 \leq x \leq 4$.
Both: -3. So x must be greater than -3 and at most 4.

b. The quadratic $(x - 1)(x - 4)$ is negative between its roots. $(x - 1)(x - 4)$ gives 1.
$3x > 3$ gives $x > 1$. Both: 1. Here the linear condition is already satisfied by every x in 1, so the solution is 1.

Answer to Question 7

a. Set the costs of A and B equal, with m the number of months. $25 + 18m = 23m$, so $5m = 25$ and $m =$ **5 months** (both cost £115).

b. Substitute $m = 12$ into each cost formula. A: $25 + 18 \times 12 = £241$. B:
$23 \times 12 = £276$. C: $60 + 14 \times 12 = £228$. For a full year gym C is the cheapest.

c. Form an inequality and round up to the next whole month. $60 + 14m$ gives 35,
$m > 8.75$. So from **month 9** gym C is cheaper than A (at 9 months: C £186, A £187).

d. Substitute $m = 6$ into each formula. 6 months: A $= 25 + 108 = £133$, B $= £138$, C

$= 60 + 84 = £144$. **Gym A** is cheapest.

Answer to Question 8

a. The perimeter is twice the sum of length and width, and the area is length times width. $2l + 2w = 34$, i.e. $l + w = 17$, and $lw = 60$.
b. Substitute $l = 17 - w$ into $lw = 60$. $l = 17 - w$, so $w(17 - w) = 60$, $w^2 - 17w + 60 = 0$, $(w - 5)(w - 12) = 0$. The garden is **12 m by 5 m**. Check: perimeter $2(12 + 5) = 34$ ✓, area $12 \times 5 = 60$ ✓.
c. Use Pythagoras. Diagonal $= \sqrt{12^2 + 5^2} = \sqrt{169} =$ **13 m**. The path is 13 m long.
d. Multiply the perimeter by the cost per metre. Perimeter 34 m, so cost $= 34 \times 22 =$ **£748**. Fencing costs £748.

Answer to Question 9

a. Profit means $P > 0$: $-0.5x^2 + 12x - 50 > 0$. Multiply by -2 (reversing the sign): $x^2 - 24x + 100$. This is the required inequality.
b. Use the quadratic formula to find the critical values, then remember that a U-shaped graph is negative between its roots. Roots: $x = \dfrac{24 \pm \sqrt{576 - 400}}{2} = 12 \pm \sqrt{44} = 12 \pm 6.633$. So 5.37.
c. Complete the square for P. $P = -0.5(x - 12)^2 + 22$. Greatest profit at $x = 12$ (1 200 loaves), $P = 22$, i.e. **£2 200**.
d. Combine the profit interval with the production limits. Allowed: $6 \leq x \leq 20$. Profit when 5.37, so within the allowed range between **600 loaves** and **1 863 loaves** (about 1 860).

Answer to Question 10

a. Total cost $= 420 + 8n \leq 1000$, so $8n \leq 580$ and $n \leq 72.5$, so $n \leq 72$ for a whole number of students.
b. Since $n > 0$, multiplying by n keeps the direction of the inequality. Cost per student $= \dfrac{420 + 8n}{n} \leq 14$. Multiply by n: $420 + 8n \leq 14n$, so $420 \leq 6n$ and $n \geq 70$.
c. Combine the two conditions $70 \leq n \leq 72$. Also $n \geq 40$ is automatically satisfied. So $n =$ **70, 71 or 72**.
d. Divide the total cost by the number of students. $n = 70$: $\dfrac{980}{70} = £14.00$. $n = 71$: $\dfrac{988}{71} = £13.92$. $n = 72$: $\dfrac{996}{72} = £13.83$. All three values are within the £14 limit.

Answer to Question 11

a. The water hits the hillside where the two heights are equal. $4x - x^2 = 0.5x$ gives $3.5x - x^2 = 0$, $x(3.5 - x) = 0$. The jet meets the hill at $x = 0$ (the nozzle) and $x = 3.5$, $y = 1.75$. Point $(3.5,\ 1.75)$.
b. Complete the square for the jet. $y = -(x - 2)^2 + 4$, so the greatest height is **4 m** at $x = 2$.
c. Complete the square for the gap. $g = -(x - 1.75)^2 + 3.0625$. Greatest gap **3.06 m** at $x = 1.75$.
d. Set the gap equal to 3 and solve the quadratic. $3.5x - x^2 = 3$ gives $x^2 - 3.5x + 3 = 0$, $(x - 1.5)(x - 2) = 0$, so $x =$ **1.5 m** or **2 m**. So the jet is 3 m above the hillside at two places.

Answer to Question 12

a. After cutting the squares, the base is $(20 - 2x)$ by $(14 - 2x)$ and the height is x. Length $= 20 - 2x$, width $= 14 - 2x$, height $= x$.
$V = x(20 - 2x)(14 - 2x) = x(280 - 68x + 4x^2) = 4x^3 - 68x^2 + 280x$.
b. Substitute $x = 2$ into the factorised form. $V(2) = 2 \times 16 \times 10 =$ **320 cm**^{3}.
c. Set $V = 288$ and divide by 4; use the factor theorem with $x = 4$.
$4x^3 - 68x^2 + 280x - 288 = 0 \Rightarrow x^3 - 17x^2 + 70x - 72 = 0$. At $x = 4$:
$64 - 272 + 280 - 72 = 0$ ✓. Divide by $(x - 4)$:
$x^3 - 17x^2 + 70x - 72 = (x - 4)(x^2 - 13x + 18)$. The quadratic gives
$x = \dfrac{13 \pm \sqrt{97}}{2} = 11.42$ or 1.58. $11.42 > 7$ is impossible (the card is only 14 cm wide), so the other solution is $x = 1.58$.
d. The width needs $14 - 2x > 0$, so 0.

Answer to Question 13

a. Let an adult ticket cost £A and a child ticket £C. $2A + 3C = 47.5$ and $3A + 2C = 52.5$.
b. Use elimination by making the coefficients of A equal. Multiply the first by 3 and the second by 2: $6A + 9C = 142.5$ and $6A + 4C = 105$. Subtract: $5C = 37.5$, $C = 7.5$. Then $2A = 47.5 - 22.5 = 25$, $A = 12.5$. Adult **£12.50**, child **£7.50**. Check: $3 \times 12.5 + 2 \times 7.5 = 52.5$ ✓.
c. Substitute the prices. $2 \times 12.50 + 4 \times 7.50 = 25 + 30 =$ **£55**.
d. Compare the family ticket with the normal price. Yes, the family ticket saves £55 − £50 = **£5**.

Answer to Question 14

a. Substitute the speeds into the formula. $v = 30$: $0.02 \times 900 + 18 = 36$ ✓. $v = 50$: $0.02 \times 2500 + 30 = 80$ ✓. Both entries are correct.
b. Set $d \leq 60$ and clear the decimals. $0.02v^2 + 0.6v \leq 60$. Multiply by 50: $v^2 + 30v \leq 3000$, i.e. $v^2 + 30v - 3000 \leq 0$. This is the required inequality.
c. Use the quadratic formula; a U-shaped graph is negative between its roots. $v = \dfrac{-30 \pm \sqrt{900 + 12000}}{2} = \dfrac{-30 \pm 113.58}{2}$, so $v = 41.79$ or -71.79. The inequality holds between the roots, so $v \leq$ **41.8 mph**.
d. A speed cannot be negative. So only the positive root gives the maximum speed of 41.8 mph.

Answer to Question 15

a. $150a + 130c = 3270$ and $200a + 120c = 3880$.
b. Use elimination by making the coefficients of a equal. Multiply the first by 4 and the second by 3: $600a + 520c = 13080$ and $600a + 360c = 11640$. Subtract: $160c = 1440$, $c = 9$. Then $150a = 3270 - 1170 = 2100$, $a = 14$. Adults **£14**, children **£9**. Check: $200 \times 14 + 120 \times 9 = 2800 + 1080 = 3880$ ✓.
c. Write the takings in terms of the number of adult tickets, then use the restriction on child tickets. Let x be the number of adult tickets; then children $= 400 - x \geq 150$, so $x \leq 250$. Takings $= 14x + 9(400 - x) = 5x + 3600$, greatest when $x = 250$: $5 \times 250 + 3600 =$ **£4 850** (250 adults and 150 children: $3500 + 1350$).
d. Each adult ticket brings £14 but costs £2.50, so the fixed cost of £2 900 must be covered. Profit per ticket (all adult) $= 14 - 2.5 = £11.50$. Need $11.5n > 2900$, so $n > 252.17$. At least **253 tickets**.

Answer to Question 16

a. Set the two expressions for y equal. $x^2 - 4x + 1 = x + k \Rightarrow x^2 - 5x + 1 - k = 0$.
b. Two distinct points need a positive discriminant. Discriminant $= 25 - 4(1 - k) = 21 + 4k > 0$, so $k > -\frac{21}{4}$.
c. Substitute $k = -5$ and factorise. $x^2 - 5x + 6 = 0$, $(x - 2)(x - 3) = 0$. Points $(2, -3)$ and $(3, -2)$ (using $y = x - 5$). Distance $= \sqrt{1^2 + 1^2} = \sqrt{2} \approx$ **1.41**.
d. A tangent meets the curve at exactly one point, so the discriminant is zero. Tangent when $21 + 4k = 0$, $k = -\frac{21}{4}$. Then $x^2 - 5x + \frac{25}{4} = 0$, $(x - \frac{5}{2})^2 = 0$, $x = \frac{5}{2}$, $y = \frac{5}{2} - \frac{21}{4} = -\frac{11}{4}$. Point $(\frac{5}{2}, -\frac{11}{4})$.

Answer to Question 17

a. Use the factor theorem, then divide by $(x - 3)$ and factorise the quadratic

quotient. $f(3) = 54 - 27 - 33 + 6 = 0$, so $(x - 3)$ is a factor. Divide: $f(x) = (x - 3)(2x^2 + 3x - 2) = (x - 3)(2x - 1)(x + 2)$.

b. Use the roots, the y-intercept and the sign of the leading coefficient. Roots at $x = -2,\ \frac{1}{2},\ 3$; y-intercept $f(0) = 6$; positive x^3 coefficient so the curve rises on the right and falls on the left (as in the figure).

c. Read from the sketch where the curve is above the x-axis. $f(x) > 0$ for -2 **or** $x > 3$.

d. Use the remainder theorem with $a = -1$. $f(-1) = -2 - 3 + 11 + 6 =$ **12**. The remainder is 12.

Answer to Question 18

a. Substitute $x = 4$: the length is 8 m and the width is 3 m. $x = 4$: $8 \times 3 = 24$ and $2(8 + 3) = 22$ ✓.

b. Expand the area and subtract 50. $(x + 4)(x - 1)$. $(x + 9)(x - 6)$, so -9.

c. The perimeter is twice the sum of length and width. Perimeter $= 2[(x + 4) + (x - 1)] = 2(2x + 3) = 4x + 6 \geq 20$, so $x \geq 3.5$.

d. Both: $3.5 \leq x$ (and $x > 1$ for a positive width). Whole numbers: $x = 4$ (8 m by 3 m) and $x = 5$ (9 m by 4 m).

Answer to Question 19

a. Expand the factorised form. $x(x - 3)^2 = x(x^2 - 6x + 9) = x^3 - 6x^2 + 9x$. The root at $x = 3$ is repeated, so the track **touches** the ground level at $(3, 0)$ without crossing it.

b. Set $x^3 - 6x^2 + 9x = 4$: $x^3 - 6x^2 + 9x - 4 = 0$. Expand $(x - 1)^2(x - 4) = (x^2 - 2x + 1)(x - 4) = x^3 - 6x^2 + 9x - 4$ ✓. Beam meets the track at $x = 1$ (touching) and $x = 4$.

c. A repeated root means the beam is a **tangent** at $x = 1$: the track reaches its first peak at height 4 and just touches the beam there before descending.

d. $x(x - 3)^2 > 0$ for $x > 0$ with $x \neq 3$. So above ground for 0 and 3.

Answer to Question 20

a. Use the factor theorem twice to obtain two equations in a and b.
$p(2) = 16 + 8a - 28 + 2b + 6 = 0 \Rightarrow 4a + b = 3$.
$p(-3) = 81 - 27a - 63 - 3b + 6 = 0 \Rightarrow 9a + b = 8$. Subtract: $5a = 5$, $a =$ **1**, $b = -1$.

b. $p(x) = x^4 + x^3 - 7x^2 - x + 6$. Divide by $(x - 2)(x + 3) = x^2 + x - 6$: quotient $x^2 - 1$. So $p(x) = (x - 2)(x + 3)(x - 1)(x + 1)$.

c. Roots at $-3, -1, 1, 2$ and the quartic is positive for large $|x|$. Sign pattern: positive for x, negative for -3, positive for -1, negative for 1, positive for $x > 2$. So $p(x)$ when -3 **or** 1.

d. Use the remainder theorem with $a = 3$. $p(3) = 81 + 27 - 63 - 3 + 6 = \mathbf{48}$. The remainder is 48.