Worksheet - Proof

Year: Year 12 · 21

Show your working.

Question 1

Question 1 – Even and odd algebra *(Section A · Skills practice)*

Let n be any integer. Even integers can be written as $2n$ and odd integers as $2n + 1$.

a. Prove that the sum of any two odd numbers is even. [3 marks]
b. Prove that the product of an even number and any integer is even. [2 marks]
c. Prove that the square of an odd number is odd. [3 marks]

Question 2

Question 2 – Consecutive integers *(Section A · Skills practice)*

Three consecutive integers can be written n, $n + 1$ and $n + 2$.

a. Prove that the sum of three consecutive integers is a multiple of 3. [2 marks]
b. Prove that the sum of four consecutive integers is never a multiple of 4. [3 marks]
c. Prove that the product of two consecutive integers is always even. [3 marks]

Question 3

Question 3 – Disproof by counter-example *(Section A · Skills practice)*

Find a counter-example to show that each statement is false.

a. "$n^2 + n + 11$ is prime for every positive integer n." [2 marks]
b. "If $x^2 > 9$ then $x > 3$." [2 marks]
c. "The sum of two prime numbers is always even." [2 marks]
d. "$\sqrt{a^2 + b^2} = a + b$ for all positive a and b." [2 marks]

Question 4

Question 4 – Proof by exhaustion *(Section A · Skills practice)*

Use exhaustion to prove each statement.

a. The square of any integer ends in 0, 1, 4, 5, 6 or 9. [3 marks]
b. Every cube of an integer is either a multiple of 9, or one more than a multiple of 9, or one less than a multiple of 9. [4 marks]

Question 5

Question 5 – Algebraic identities *(Section A · Skills practice)*

Prove each identity.

a. $(n+3)^2-(n-3)^2 \equiv 12n$ [3 marks]
b. $(2n+1)^2-(2n-1)^2 \equiv 8n$ [3 marks]
c. $n^3-n \equiv n(n-1)(n+1)$ [3 marks]

Question 6

Question 6 – Proof by contradiction *(Section A · Skills practice)*

To prove $\sqrt{2}$ is irrational, assume the opposite and look for a contradiction.

a. Assume $\sqrt{2}=\frac{a}{b}$ where $\frac{a}{b}$ is a fraction in its simplest form. Show that $a^2=2b^2$. [1 mark]
b. Explain why a must be even. [2 marks]
c. Write $a=2k$, show that b is also even, and state the contradiction. [3 marks]

Question 7

Question 7 – The calendar square *(Section B · Real-life practice)*

A kitchen calendar shows each month in rows of 7 days. Amira draws a square around four numbers in a 2×2 block, for example 9, 10, 16 and 17. She notices that $9+17=26$ and $10+16=26$. She wonders whether this always happens.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21

a. Let the top-left number be n. Write the other three numbers of the block in terms of n. [2 marks]
b. Prove that the two diagonal sums are always equal. [2 marks]
c. Prove that the sum of the four numbers is always a multiple of 4. [2 marks]
d. Amira says the total of the block could be 70. Is she right? [2 marks]

Question 8

Question 8 – The garden path *(Section B · Real-life practice)*

A square lawn has side x metres, where x is a whole number. A path of width 1 m is laid around all four sides, so the lawn plus path is a larger square. The path is covered with paving slabs.

a. Show that the area of the path is $4x + 4$ square metres. [3 marks]
b. Prove that the area of the path is always a multiple of 4. [2 marks]
c. The area of the path is 84 m^2. Find x. [2 marks]
d. Explain why the path area can never be an odd number. [1 mark]

Question 9

Question 9 – Handshakes at the school reunion *(Section B · Real-life practice)*

At a school reunion, n people each shake hands once with every other person.

a. Show that the total number of handshakes is $\frac{n(n-1)}{2}$. [2 marks]
b. Prove that $n(n-1)$ is always even, so the formula always gives a whole number. [3 marks]
c. There were 66 handshakes. How many people attended? [3 marks]
d. Can the number of handshakes ever be 50? Justify your answer. [2 marks]

Question 10

Question 10 – Divisibility of phone-number games *(Section B · Real-life practice)*

Ben claims: "Take any two-digit number, reverse its digits, and add the two numbers. The answer is always a multiple of 11." A two-digit number with tens digit a and units digit b equals $10a + b$.

a. Test Ben\'s claim with 37 and 82. [2 marks]
b. Prove the claim. [3 marks]
c. Prove that the difference between a two-digit number and its reverse is always a multiple of 9. [3 marks]
d. Is the difference always a multiple of 11? Give a counter-example if not. [2 marks]

Question 11

Question 11 – Tiling with square tiles *(Section B · Real-life practice)*

A tiler arranges square tiles in a square pattern of n by n tiles. She then adds a border of tiles to make an $(n+1)$ by $(n+1)$ square. She claims the number of extra tiles is always odd.

a. Show that the number of extra tiles is $2n + 1$. [2 marks]
b. Prove her claim. [2 marks]
c. The extra tiles cost £0.85 each and she has a £30 budget. What is the largest square she can extend? [3 marks]
d. Prove that the sum of the first n odd numbers is n^2 using the tile pattern or by algebra. [3 marks]

Question 12

Question 12 – Taxi fares and parity *(Section B · Real-life practice)*

A taxi company charges £3 plus £2 per kilometre (whole kilometres only). A bus company charges £5 plus £4 per kilometre.

a. Show that the taxi fare for n km is always an odd number of pounds. [2 marks]
b. Show that the bus fare is always odd. [2 marks]
c. Prove that the difference between the two fares is always even. [2 marks]
d. For which distance are the two fares equal? Explain. [2 marks]

Question 13

Question 13 – Number tricks with age *(Section B · Real-life practice)*

A magician asks a volunteer: "Think of a whole number. Double it, add 6, halve it, then subtract your original number. The answer is 3."

a. Show algebraically that the answer is always 3. [3 marks]
b. The volunteer instead starts with n, squares it, adds n, and divides by n (with $n \neq 0$), then subtracts n. Show the answer is always 1. [3 marks]
c. The magician asks: "Pick a three-digit number, repeat it to make a six-digit number (such as 247247). It is always divisible by 7, 11 and 13." Prove it. [4 marks]

Question 14

Question 14 – Savings jar patterns *(Section B · Real-life practice)*

Jaya puts £1 in a jar on day 1, £2 on day 2, £3 on day 3 and so on, adding one pound more each day.

a. Show that after n days she has saved £$\frac{1}{2}n(n+1)$. [2 marks]
b. Prove that after n days the amount is a whole number of pounds. [2 marks]
c. On which day does her total first exceed £500? [3 marks]
d. Prove that the total after n days plus the total after $n+1$ days is a square number. [3 marks]

Question 15

Question 15 – Pythagorean triples *(Section C · Problem solving)*

A Pythagorean triple is a set of three positive integers (a, b, c) with $a^2 + b^2 = c^2$. A carpenter uses the 3-4-5 triangle to check that corners are square. Jess finds a rule: for any positive integer n, the numbers $2n+1$, $2n^2+2n$ and $2n^2+2n+1$ form a triple.

n	$2n+1$	$2n^2+2n$	$2n^2+2n+1$

n	$2n+1$	$2n^2+2n$	$2n^2+2n+1$
1	3	4	5
2	5	12	13
3	7	24	25

a. Check the rule for $n = 4$. [2 marks]
b. Prove the rule works for every positive integer n. [4 marks]
c. Use the rule to find a triple with the longest side 85. [3 marks]
d. Show that the carpenter\'s triple can be scaled to give a triangle with perimeter 60 m. [2 marks]

Question 16

Question 16 – Primes and the infinite list *(Section C · Problem solving)*

Euclid proved that there are infinitely many prime numbers. His proof is a famous proof by contradiction. Suppose there is a finite list of all primes $p_1, p_2, \ldots, p_k$. Let $N = p_1 p_2 \cdots p_k + 1$.

a. Explain why N leaves remainder 1 when divided by each prime p_i. [2 marks]
b. Explain why N is either prime or has a prime factor not on the list. [3 marks]
c. State the contradiction and conclude. [2 marks]
d. Calculate N for the list 2, 3, 5, 7, 11, 13 and show that N is not prime. Does this break the argument? Explain. [3 marks]

Question 17

Question 17 – The sum of squares problem *(Section C · Problem solving)*

Four friends compare their results on a number puzzle. Each claims a statement about integers n. For each statement decide whether it is true or false. If true, prove it. If false, give a counter-example.

a. "$n^2 - n$ is always even." [3 marks]
b. "$n^2 + n + 1$ is always odd." [3 marks]
c. "$n^3 + n$ is always divisible by 3." [3 marks]
d. "$2^n + 1$ is prime for all $n \geq 1$." [3 marks]

Question 18

Question 18 – Mean of consecutive terms *(Section C · Problem solving)*

A teacher claims some facts about means. Let n be a positive integer.

a. Prove that the mean of three consecutive integers is the middle integer. [2 marks]
b. Prove that the sum of the squares of three consecutive integers is always 2 more

than a multiple of 3. [3 marks]

c. Prove that the sum of the squares of two consecutive integers is always odd. [3 marks]

d. Find two consecutive integers whose squares add to 421. [3 marks]

Question 19

Question 19 – Proof with inequalities *(Section C · Problem solving)*

Prove each inequality for all real values stated.

a. $x^2 - 6x + 10 > 0$ for all real x. [3 marks]

b. $a^2 + b^2 \geq 2ab$ for all real a and b. [3 marks]

c. $x + \dfrac{1}{x} \geq 2$ for all $x > 0$. [4 marks]

d. Hence show that the sum of a positive number and its reciprocal is never less than 2, and state when it equals 2. [2 marks]

Question 20

Question 20 – Exhaustion with a table *(Section C · Problem solving)*

Prove that no square number can be written in the form $4k + 2$ or $4k + 3$, where k is a non-negative integer. A student starts to fill in a table.

Form of n	n^2	Remainder when n^2 is divided by 4
$2m$	$4m^2$	0
$2m + 1$	$4m^2 + 4m + 1$	1

a. Explain why every integer has one of the two forms in the table. [1 mark]

b. Complete the proof. [3 marks]

c. Use your result to show that 1 000 003 is not a perfect square. [2 marks]

d. Show that $x^2 - 4y = 3$ has no integer solutions. [3 marks]

Question 21

Question 21 – Challenge: irrational sums *(Section D · Challenge)*

Use proof by contradiction to answer this question.

a. Prove that if x is rational and y is irrational then $x + y$ is irrational. [4 marks]

b. Prove that $\sqrt{2} + \sqrt{3}$ is irrational, given that $\sqrt{6}$ is irrational. [5 marks]

c. Prove that $\log_2 3$ is irrational. [4 marks]