Answers - Proof

Year: Year 12

Answer to Question 1

a. Let the odd numbers be $2m+1$ and $2n+1$. Their sum is $2m+2n+2=2(m+n+1)$. This is 2 times an integer, so it is **even**.
b. Let the even number be $2m$ and the other integer be n. The product is $2mn=2(mn)$, which is 2 times an integer, so it is **even**.
c. $(2n+1)^2=4n^2+4n+1=2(2n^2+2n)+1$. This has the form $2k+1$ with $k=2n^2+2n$ an integer, so the square is **odd**.

Answer to Question 2

a. $n+(n+1)+(n+2)=3n+3=3(n+1)$, a multiple of 3.
b. $n+(n+1)+(n+2)+(n+3)=4n+6=4(n+1)+2$. This leaves remainder 2 when divided by 4, so it is never a multiple of 4.
c. The product is $n(n+1)$. Either n is even or $n+1$ is even (one of two consecutive integers is even). An even number times an integer is even, so the product is even. Alternatively, check both cases: if $n=2k$ the product is $2k(2k+1)$, and if $n=2k+1$ it is $(2k+1)(2k+2)=2(2k+1)(k+1)$. Both are even.

Answer to Question 3

a. Try $n=1$: 13, prime. $n=2$: 17, prime. $n=3$: 23. $n=4$: 31. $n=5$: 41. $n=6$: 53. $n=7$: 67. $n=8$: 83. $n=9$: 101. $n=10$: $121=11^2$, which is **not prime**. So $n=10$ is a counter-example.
b. Take $x=-4$. Then $x^2=16>9$ but -4 is not greater than 3. **Counter-example: $x=-4$.**
c. $2+3=5$ is odd (and 5 is not even). **Counter-example: 2 and 3.**
d. Take $a=3$ and $b=4$: $\sqrt{9+16}=5$ but $a+b=7$. **Counter-example: $a=3$, $b=4$.**

Answer to Question 4

a. The last digit of n^2 depends only on the last digit of n. Last digits 0 to 9 give squares ending in $0,1,4,9,6,5,6,9,4,1$. Every one of these is in the list {0, 1, 4, 5, 6, 9}, so the statement is true.
b. Every integer can be written $3k$, $3k+1$ or $3k+2$. $(3k)^3=27k^3=9(3k^3)$. $(3k+1)^3=27k^3+27k^2+9k+1=9(3k^3+3k^2+k)+1$.

$(3k+2)^3 = 27k^3 + 54k^2 + 36k + 8 = 9(3k^3 + 6k^2 + 4k) + 9 - 1 = 9(3k^3 + 6k^2 + 4k + 1) - 1$. These three cases cover every integer, so the statement is true.

Answer to Question 5

a. To prove an identity, start from one side and simplify until you reach the other side. Expand both squares using $(a \pm b)^2 = a^2 \pm 2ab + b^2$. LHS $= (n^2 + 6n + 9) - (n^2 - 6n + 9) = 12n =$ RHS. The n^2 and constant terms cancel, which proves the identity for every value of n.
b. Use the same method, expanding $(2n \pm 1)^2$. LHS $= (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n =$ RHS. So the difference of the squares of two consecutive odd numbers is always a multiple of 8.
c. Here it is easier to start from the right-hand side and expand. RHS $= n(n^2 - 1) = n^3 - n =$ LHS. (Factorising the LHS: $n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$ by the difference of two squares.) Since $n-1$, n and $n+1$ are three consecutive integers, this also shows that $n^3 - n$ is always divisible by 6.

Answer to Question 6

a. Squaring gives $2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$.
b. $a^2 = 2b^2$ is even, and the square of an odd number is odd, so a cannot be odd. Hence a is **even**.
c. $a = 2k$ gives $4k^2 = 2b^2$, so $b^2 = 2k^2$, which is even, so b is even by the same argument. Now a and b are both even, so the fraction $\frac{a}{b}$ was not in its simplest form. This **contradicts** the assumption, so $\sqrt{2}$ is irrational.

Answer to Question 7

a. The block is n, $n+1$, $n+7$, $n+8$.
b. Diagonals: $n + (n+8) = 2n + 8$ and $(n+1) + (n+7) = 2n + 8$. They are equal.
c. Total $= n + (n+1) + (n+7) + (n+8) = 4n + 16 = 4(n+4)$. So the total is **always a multiple of 4**. For example $9 + 10 + 16 + 17 = 52 = 4 \times 13$.
d. $4n + 16 = 70$ gives $4n = 54$, $n = 13.5$, not an integer. So **no**: 70 is not a multiple of 4, so it cannot be the total.

Answer to Question 8

a. The path is the region between the larger square and the lawn. Large square has

side $x+2$, so area $(x+2)^2 = x^2 + 4x + 4$. Subtract the lawn x^2: path area $= 4x + 4$. This is the required expression.

b. Factorise the expression. $4x + 4 = 4(x+1)$, which is 4 times an integer. So the area of the path is always a multiple of 4.

c. Form an equation and solve it. $4x + 4 = 84$ gives $x = 20$. The lawn is 20 m by 20 m; check: $22^2 - 20^2 = 84$ ✓.

d. $4(x+1)$ is always even (a multiple of 4), so it can never be odd. So the path area is never odd.

Answer to Question 9

a. Each of n people shakes hands with $n-1$ others, giving $n(n-1)$, but every handshake has been counted twice (once by each person). So the total is $\frac{n(n-1)}{2}$.

b. If n is even then $n(n-1)$ is even. If n is odd then $n-1$ is even and the product is even. Either way the product is even, so dividing by 2 gives an integer.

c. $n(n-1) = 132$, so $n^2 - n - 132 = 0$, $(n-12)(n+11) = 0$. Since $n > 0$, $n =$ **12**.

d. $n(n-1) = 100$. $10 \times 9 = 90$ and $11 \times 10 = 110$, so no integer works. **No.**

Answer to Question 10

a. Reverse the digits of each number and add. $37 + 73 = 110 = 11 \times 10$. $82 + 28 = 110 = 11 \times 10$. Both work. Both examples support the claim, but examples alone do not prove it.

b. The reversed number has tens digit b and units digit a, so it equals $10b + a$. $(10a + b) + (10b + a) = 11a + 11b = 11(a+b)$, a multiple of 11. Since $a + b$ is an integer, the sum is always a multiple of 11.

c. Subtract instead of adding. $(10a + b) - (10b + a) = 9a - 9b = 9(a-b)$, a multiple of 9. Since $a - b$ is an integer, the difference is always a multiple of 9.

d. To show that a statement is false, one counter-example is enough. For 52 and 25: $52 - 25 = 27$, and 27 is not a multiple of 11. **Counter-example: 52.** So the difference is not always a multiple of 11.

Answer to Question 11

a. $(n+1)^2 - n^2 = 2n + 1$.

b. $2n + 1 = 2(n) + 1$ has the form $2k + 1$, so it is always odd.

c. $0.85(2n+1) \leq 30$ gives $2n + 1 \leq 35.29$, so $2n + 1 \leq 35$, $n \leq 17$. So the largest starting square is 17×17 extended to 18×18.

d. Going from $k \times k$ to $(k+1) \times (k+1)$ adds the odd number $2k + 1$. Starting with 1 tile, adding 3, then 5, and so on gives $1 + 3 + 5 + \cdots + (2n-1) = n^2$. Algebraically, the sum of an arithmetic series with n terms, first term 1 and last

term $2n-1$, is $\frac{n}{2}(1+2n-1)=n^2$.

Answer to Question 12

a. Write the fare as $3+2n$ and look for the form $2k+1$. Fare $=3+2n=2n+2+1=2(n+1)+1$, odd. So every taxi fare is an odd number of pounds.
b. Write the fare as $5+4n$. Fare $=5+4n=4n+4+1=2(2n+2)+1$, odd. So every bus fare is also an odd number of pounds.
c. Subtract the two fares and factorise. Odd minus odd is even: $(5+4n)-(3+2n)=2n+2=2(n+1)$. Since $n+1$ is an integer, the difference is always even.
d. Set the fares equal, which is the same as setting the difference equal to 0. $2n+2=0$ gives $n=-1$, which is impossible, so the fares are **never equal** (the bus is always dearer). The difference is always at least £2.

Answer to Question 13

a. Let the number be n and write each step algebraically. $\dfrac{2n+6}{2}-n=n+3-n=3$. The original number cancels, so the answer is always 3, whatever number is chosen.
b. Write each step algebraically; since $n\neq 0$ we can divide by n. $\dfrac{n^2+n}{n}-n=n+1-n=1$. The answer is always 1.
c. Writing a three-digit number twice is the same as multiplying it by 1000 and adding the number itself. A three-digit number N repeated is $1000N+N=1001N$. $1001=7\times 11\times 13$, so $1001N$ is divisible by 7, 11 and 13. For example $247247=1001\times 247$.

Answer to Question 14

a. The total is $1+2+\cdots+n=\dfrac{n}{2}(1+n)$, the sum of an arithmetic series.
b. $n(n+1)$ is a product of consecutive integers so it is even, and dividing by 2 gives an integer.
c. $\frac{1}{2}n(n+1)>500$ gives $n(n+1)>1000$. $31\times 32=992$ and $32\times 33=1056>1000$. So on **day 32** (total £528).
d. $\frac{1}{2}n(n+1)+\frac{1}{2}(n+1)(n+2)=\frac{1}{2}(n+1)(n+n+2)=(n+1)^2$.

Answer to Question 15

a. $n = 4$: 9, 40, 41. $9^2 + 40^2 = 81 + 1600 = 1681 = 41^2$. It works.
b. $(2n+1)^2 + (2n^2+2n)^2 = 4n^2 + 4n + 1 + 4n^4 + 8n^3 + 4n^2 = 4n^4 + 8n^3 + 8n^2 + 4n + 1$. And $(2n^2+2n+1)^2 = 4n^4 + 8n^3 + 4n^2 + 4n^2 + 4n + 1 = 4n^4 + 8n^3 + 8n^2 + 4n + 1$. They are equal, so the rule always works.
c. $2n^2 + 2n + 1 = 85$ gives $n^2 + n - 42 = 0$, $(n+7)(n-6) = 0$, $n = 6$. The triple is 13, 84, 85: $169 + 7056 = 7225 = 85^2$.
d. The 3-4-5 triangle has perimeter 12. Multiply every side by 5 to get 15, 20, 25 with perimeter 60.

Answer to Question 16

a. The product $p_1 p_2 \cdots p_k$ is divisible by every p_i, so adding 1 gives remainder 1 on division by each p_i.
b. $N > 1$, so it is either prime itself (and N is bigger than every prime on the list, so it is a new prime) or it has a prime factor. That prime factor cannot be any p_i because dividing by p_i leaves remainder 1. So there is a prime not on the list.
c. The list was assumed to contain **all** primes, but we have found a prime that is not on it. This is a contradiction, so the assumption is false and there are infinitely many primes.
d. $N = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$. So N is **not** prime. This does not break the argument: the prime factors 59 and 509 are not on the list, which is exactly what the proof predicts.

Answer to Question 17

a. Factorise the expression. **True.** $n^2 - n = n(n-1)$, a product of consecutive integers, which is always even. So the statement is true for every integer n.
b. Use the result of part (a) with $n^2 + n$. **True.** $n^2 + n = n(n+1)$ is even, so adding 1 gives an odd number. So the statement is true for every integer n.
c. Try small values of n: $n = 1$ gives 2, which is not a multiple of 3 either. **False.** $n = 2$ gives $8 + 2 = 10$, not a multiple of 3. So the statement is false.
d. Try small values: $n = 1$ gives 3 and $n = 2$ gives 5, both prime. **False.** $n = 3$ gives $2^3 + 1 = 9 = 3 \times 3$, which is not prime. So the statement is false.

Answer to Question 18

a. Write the three integers as $n-1$, n and $n+1$. $\dfrac{(n-1) + n + (n+1)}{3} = \dfrac{3n}{3} = n$.
So the mean is the middle integer.
b. Expand each square and collect terms.
$(n-1)^2 + n^2 + (n+1)^2 = (n^2 - 2n + 1) + n^2 + (n^2 + 2n + 1) = 3n^2 + 2$, which is

2 more than the multiple of 3 given by $3n^2$. So the sum always leaves remainder 2 when divided by 3.

c. Write the integers as n and $n+1$. $n^2+(n+1)^2 = 2n^2+2n+1 = 2(n^2+n)+1$, odd. So the sum is always odd.

d. Use the expression from part (c) and solve the quadratic. $2n^2+2n+1 = 421$ gives $n^2+n-210 = 0$, $(n+15)(n-14) = 0$, so $n = 14$. The integers are 14 and 15: $196+225 = 421$. Using the negative root $n = -15$ gives the pair -15 and -14, which also works.

Answer to Question 19

a. $x^2-6x+10 = (x-3)^2+1$. A square is at least 0, so the expression is at least 1, which is positive.

b. $a^2+b^2-2ab = (a-b)^2 \geq 0$, so $a^2+b^2 \geq 2ab$.

c. Since $x > 0$, multiply by x without reversing the inequality: $x^2+1 \geq 2x$, i.e. $x^2-2x+1 \geq 0$, i.e. $(x-1)^2 \geq 0$, which is true. Reversing the steps proves the original.

d. The inequality in part (c) is exactly this statement. Equality holds when $(x-1)^2 = 0$, i.e. $x = 1$.

Answer to Question 20

a. Every integer is even ($2m$) or odd ($2m+1$).

b. n^2 leaves remainder 0 or 1 on division by 4. The remainders 2 and 3 never occur, so a square cannot have the form $4k+2$ or $4k+3$.

c. $1\,000\,003 = 4 \times 250\,000 + 3$, which has the form $4k+3$, so it is not a square.

d. $x^2 = 4y+3$ would mean x^2 has remainder 3 on division by 4, which is impossible. So there are no integer solutions.

Answer to Question 21

a. Assume $x+y = r$ is rational. Then $y = r-x$, a difference of two rational numbers, which is rational. This contradicts y being irrational. So $x+y$ is irrational.

b. Assume $\sqrt{2}+\sqrt{3} = r$ is rational. Squaring gives $5+2\sqrt{6} = r^2$, so $\sqrt{6} = \dfrac{r^2-5}{2}$, which is rational. This contradicts the given fact that $\sqrt{6}$ is irrational. So $\sqrt{2}+\sqrt{3}$ is irrational.

c. Assume $\log_2 3 = \dfrac{p}{q}$ with positive integers p, q. Then $2^{p/q} = 3$, so $2^p = 3^q$. The left side is even and the right side is odd (for $q \geq 1$), a contradiction. So $\log_2 3$ is irrational.