Test answers - Probability and the binomial distribution

Year: Year 12 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. Rearrange the addition rule: $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.
$0.6 + 0.5 - 0.8 = 0.3$ (2) So $P(A \cap B) = 0.3$. Since $P(A \cap B) > 0$, the events can happen together, so they are not mutually exclusive.

b. Use $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$ and compare with $P(B)$. $P(B|A) = \dfrac{0.3}{0.6} = 0.5 = P(B)$ (1), so independent (1) Equivalently $P(A)P(B) = 0.6 \times 0.5 = 0.3 = P(A \cap B)$. So A and B are independent: knowing that A happened does not change the probability of B.

The second mark in (b) requires the comparison with $P(B)$, not just the word 'independent'.

Answer to Question 2 (4 marks)

Same colour means both white or both black. Without replacement the second probabilities change, so multiply along each branch and add the two paths.

$P(WW) = \dfrac{4}{10} \times \dfrac{3}{9} = \dfrac{12}{90}$ (1). $P(BB) = \dfrac{6}{10} \times \dfrac{5}{9} = \dfrac{30}{90}$ (1). Sum $= \dfrac{42}{90}$ (1) $= \dfrac{7}{15}$ (1).

The probability that the two balls are the same colour is $\dfrac{7}{15} \approx 0.467$.

Alternatively, $1 - P(\text{different colours}) = 1 - 2 \times \dfrac{4}{10} \times \dfrac{6}{9} = 1 - \dfrac{48}{90} = \dfrac{42}{90}$ ✓.

Using $\frac{4}{10} \times \frac{4}{10}$ (with replacement) is the most frequent error and scores only the addition mark.

Answer to Question 3 (4 marks)

Use $P(X = r) = \binom{10}{r}(0.2)^r(0.8)^{10-r}$. For (b), use the complement:
$P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$.

(a) $\binom{10}{3}(0.2)^3(0.8)^7 = 0.2013$ (2). (b)
$1 - 0.8^{10} - 10(0.2)(0.8)^9 = 1 - 0.1074 - 0.2684 = 0.6242$ (2).

So $P(X = 3) \approx 0.201$ and $P(X \geq 2) \approx 0.624$.

In (a), $\binom{10}{3} = 120$, so the probability is $120 \times 0.008 \times 0.2097$. In (b), the complement avoids adding nine separate probabilities.

The binomial PD and CD functions of a calculator give the same values. Writing $P(X \geq 2) = 1 - P(X \leq 2)$ is a common mistake that wrongly removes $P(X = 2)$. Award 1 mark for the method and 1 for the value in each part.

Answer to Question 4 (4 marks)

a. There are 50 independent tosses with constant probability 0.5; use np and $np(1-p)$. $X \sim B(50, 0.5)$, mean 25, variance 12.5 (2) The standard deviation is $\sqrt{12.5} \approx 3.54$. The distribution is symmetrical about the mean 25.

b. Use the binomial formula with $r = 25$. $\binom{50}{25} 0.5^{50} = 0.1123$ (2) Even the most likely number of heads has only about an 11% chance.

No normal approximation is needed here: the exact binomial value is expected.

Answer to Question 5 (4 marks)

a. At least once is the complement of never: $P(\text{no blue}) = 0.85^{12}$. $1 - 0.85^{12} = 0.858$ (2) The probability is about 0.858.

b. The number of blues is $B(12, 0.15)$; use the binomial formula with $r = 2$. $\binom{12}{2}(0.15)^2(0.85)^{10} = 0.2924$ (1) The probability is about 0.292.

c. Solve $1 - 0.85^n > 0.95$ with logarithms; dividing by $\ln 0.85 < 0$ reverses the inequality. $0.85^n < 0.05$, $n > 18.4$, so 19 (1) At least 19 spins are needed.

Check (c): $0.85^{18} \approx 0.0536$ is too large, but $0.85^{19} \approx 0.0456 < 0.05$ ✓.