Lesson - Integration

Year: Year 12

Integration is the reverse of differentiation. It is used to recover a function from its gradient and to find areas under curves. Learn to integrate powers of x, to find the constant of integration, and to evaluate definite integrals accurately.

Starter activity

A tap fills a bucket at the rate of 2 litres per minute for 5 minutes. How much water is in the bucket? Now the tap is slowly turned off so that the rate falls steadily from 2 litres per minute to 0 over 5 minutes. Sketch the rate against time. How could you use the shape of the graph to find the amount of water? What do you think happens to your method if the graph is a curve instead of a straight line?

Indefinite integrals

Definition: constant of integration

The c in an indefinite integral is the **constant of integration**. It is there because differentiating any constant gives 0.

Rule: integrating powers of x

If $\dfrac{dy}{dx} = x^n$ then $y = \dfrac{x^{n+1}}{n+1} + c$, for $n \neq -1$. In words: **add one to the power, divide by the new power, and add** c. Integrate term by term.

Method: integrating roots and fractions

1. Expand brackets, or divide each term of the numerator, first.
2. Rewrite roots and reciprocals as powers.
3. Integrate each term and add c.

Worked example 1

Find $\displaystyle\int (6x^2 - 4x + 3)\,dx$ **and** $\displaystyle\int \left(\sqrt{x} + \frac{3}{x^2}\right) dx$**.**

The first is $2x^3 - 2x^2 + 3x + c$. For the second, $\displaystyle\int \left(x^{1/2} + 3x^{-2}\right) dx = \frac{2}{3}x^{3/2} - 3x^{-1} + c$.

Worked example 2

Find $\int \frac{x^2+5}{x^2}\,dx$**.**

Divide first: $\int (1+5x^{-2})\,dx = x - \frac{5}{x} + c$.

Finding the constant

Worked example 3

Given $\frac{dy}{dx} = 3x^2 - 4$ **and that the curve passes through** $(2, 5)$**, find** y**.**

$y = x^3 - 4x + c$, so $5 = 8 - 8 + c$ and $c = 5$. The curve is $y = x^3 - 4x + 5$.

Worked example 4

Given $f'(x) = 3x^2 - 6x$ **and** $f(1) = 0$**, find** $f(2)$**.**

$f(x) = x^3 - 3x^2 + c$. Then $f(1) = 1 - 3 + c = 0$, so $c = 2$ and $f(x) = x^3 - 3x^2 + 2$. So $f(2) = 8 - 12 + 2 = -2$.

Definite integrals

Theorem: fundamental theorem of calculus

If $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$. No c is needed because it cancels.

Write the integral in square brackets with the limits.

Worked example 5

Evaluate $\int_1^3 (2x+1)\,dx$**.**

$\left[x^2 + x\right]_1^3 = (9+3) - (1+1) = 10$.

Worked example 6

Find $k > 0$ **such that** $\int_1^k 2x\,dx = 8$**.**

$\left[x^2\right]_1^k = k^2 - 1 = 8$, so $k^2 = 9$ and $k = 3$ (rejecting $k = -3$ because $k > 0$).

Area under a curve

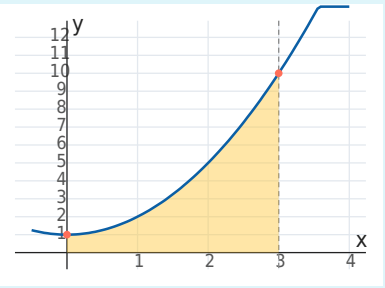
Rule: area under a curve

If $f(x) \geq 0$ on $[a, b]$, the area between the curve, the x-axis and the lines $x = a$, $x = b$ is $\int_a^b f(x)\,dx$.

Worked example 7

Find the area under $y = x^2 + 1$ from $x = 0$ to $x = 3$.

$\int_0^3 (x^2 + 1)\,dx = \left[\frac{x^3}{3} + x\right]_0^3 = 9 + 3 = 12.$



Areas below the axis and between curves

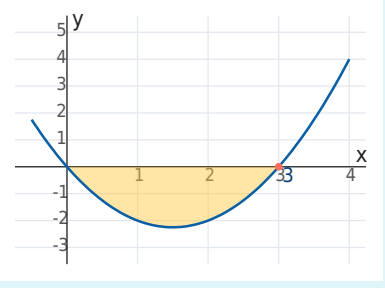
Key fact: regions below the axis

Where the curve is **below** the x-axis the integral is *negative*. The area is the positive size of that value. If the curve crosses the axis, find the roots and integrate each part separately, taking the absolute value of the negative parts.

Worked example 8

Find the area between $y = x^2 - 3x$ and the x-axis from $x = 0$ to $x = 3$.

$\int_0^3 (x^2 - 3x)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3 = 9 - 13.5 = -4.5$, so the area is 4.5.



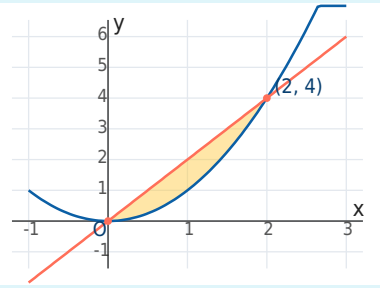
Rule: area between two curves

If $f(x) \geq g(x)$ on $[a, b]$ the area is $\int_a^b \big(f(x) - g(x)\big)\,dx$. Find the limits by solving $f(x) = g(x)$.

Worked example 9

Find the area between $y = x^2$ and $y = 2x$.

They meet where $x^2 = 2x$, so $x = 0, 2$. On $[0, 2]$, $2x \geq x^2$, so area $= \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.



Worked example 10

Find the area between $y = x^2 - 2x + 3$ and $y = x + 3$.

$x^2 - 2x + 3 = x + 3 \Rightarrow x^2 - 3x = 0$, so $x = 0$ or 3. The line is above the curve, so area $= \int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = 13.5 - 9 = 4.5$.

Note: a useful result

The area between a parabola $y = x^2 - 5x + 4$ and the x-axis between its roots 1 and 4 is $\frac{(4-1)^3}{6} = \frac{9}{2}$. Check: $\int_1^4 (x^2 - 5x + 4)\,dx = -\frac{9}{2}$, and the area is the positive value.

Watch out

Do not forget $+c$ in an indefinite integral, and divide by the new power. The order is $F(b) - F(a)$. A negative integral is not an area: take the positive value. Never integrate across a root without splitting the region. Always sketch the region first.

Worked example 11: Integrating a quotient and a negative power

Find $\int \frac{x^3 - 2x^2 + 5}{x^2}\,dx$ and evaluate it between 1 and 2.

Divide each term by x^2: $x - 2 + 5x^{-2}$. Integrate: $\frac{x^2}{2} - 2x - 5x^{-1} + c$. Between 1 and 2: at $x = 2$: $2 - 4 - 2.5 = -4.5$; at $x = 1$: $0.5 - 2 - 5 = -6.5$. The definite integral is $-4.5 - (-6.5) = 2$. Check by estimating: the integrand at $x = 1.5$ is $1.5 - 2 + 2.22 = 1.72$, so over a width of 1 the area is roughly 1.7 to 2, in line with the answer of 2.

Worked example 12: Area between a curve and the x-axis with a sign change

Find the total area between $y = x^3 - 4x$ and the x-axis for $-2 \le x \le 2$.

The curve is $x(x-2)(x+2)$, crossing at -2, 0 and 2. For -2 it is above the axis; for 0 below. $\int_{-2}^{0}(x^3 - 4x)dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} = 0 - (4 - 8) = 4$.

$\int_0^2 = \left[\frac{x^4}{4} - 2x^2\right]_0^2 = 4 - 8 = -4$. The full integral is 0, but the actual area is $4 + 4 = 8$. This is why you must split the integral where the curve crosses the axis.

Worked example 13: Using a boundary condition twice

$f''(x) = 6x - 4$, $f'(1) = 2$ and $f(1) = 3$. Find $f(x)$.

Integrate once: $f'(x) = 3x^2 - 4x + c$. With $f'(1) = 2$: $3 - 4 + c = 2$, so $c = 3$ and $f'(x) = 3x^2 - 4x + 3$. Integrate again: $f(x) = x^3 - 2x^2 + 3x + d$. With $f(1) = 3$: $1 - 2 + 3 + d = 3$, so $d = 1$. Therefore $f(x) = x^3 - 2x^2 + 3x + 1$. Check: $f'' = 6x - 4$ ✓, $f'(1) = 3 - 4 + 3 = 2$ ✓, $f(1) = 1 - 2 + 3 + 1 = 3$ ✓.

Worked example 14: Area enclosed by a curve and a chord

Find the area enclosed between $y = x^2 - 2x + 3$ and the line $y = x + 3$.

Equate: $x^2 - 2x + 3 = x + 3 \Rightarrow x^2 - 3x = 0$, so $x = 0$ and $x = 3$. The line is above the curve between them (test $x = 1$: line 4, curve 2). Area $= \int_0^3 \left[(x+3) - (x^2 - 2x + 3)\right]dx = \int_0^3 (3x - x^2)dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = 13.5 - 9 = 4.5$.

The same answer as the quadratic-and-chord rule $\frac{|a|(\beta - \alpha)^3}{6} = \frac{27}{6} = 4.5$ ✓.

Worked example 15: Linking integration with a rate of change

Oil leaks from a tank at a rate of $r = 0.4\sqrt{t}$ litres per minute. Find the total leaked in the first 9 minutes.

Total $= \int_0^9 0.4t^{1/2}dt = 0.4 \times \frac{2}{3}\left[t^{3/2}\right]_0^9 = \frac{0.8}{3} \times 27 = 7.2$ litres. The integral of a rate over a time interval always gives the total change in the quantity. The same principle

gives distance from speed, volume from flow rate and cost from marginal cost.

Worked example 16: Integrating with a context and a boundary condition

The gradient of a hillside profile is $\frac{dy}{dx} = 0.3x - 0.02x^2$ (heights in tens of metres, distance in hundreds of metres). The path starts at the origin. Find the height profile, the height at $x = 10$ and the point where the path is highest.

Integrate: $y = 0.15x^2 - \frac{0.02}{3}x^3 + c$. The path starts at the origin so $c = 0$. At $x = 10$: $y = 15 - 6.67 = 8.33$, so the height is about 83 metres. The highest point occurs where the gradient is zero: $0.3x - 0.02x^2 = x(0.3 - 0.02x) = 0$, so $x = 15$, and the height there is $0.15 \times 225 - \frac{0.02}{3} \times 3375 = 33.75 - 22.5 = 11.25$, i.e. 112.5 m. Beyond $x = 15$ the gradient is negative, so the path descends. This kind of problem combines integration with the stationary-point ideas of differentiation.

Worked example 17: Area between a curve and a horizontal line

Find the area of the region enclosed between the curve $y = x^2 - 6x + 10$ and the line $y = 5$.

Intersections: $x^2 - 6x + 10 = 5 \Rightarrow x^2 - 6x + 5 = 0 \Rightarrow (x - 1)(x - 5) = 0$, so $x = 1$ and $x = 5$. Between these values the line is above the curve (test $x = 3$: curve 1, line 5). Area $= \int_1^5 [5 - (x^2 - 6x + 10)]\,dx = \int_1^5 (-x^2 + 6x - 5)\,dx = \left[-\frac{x^3}{3} + 3x^2 - 5x\right]_1^5 = \left(-\frac{125}{3} + 75 - 25\right) - \left(-\frac{1}{3} + 3 - 5\right) = \frac{25}{3} + \frac{7}{3} = \frac{32}{3}$. The shortcut for a parabola with roots α, β gives $\frac{(\beta - \alpha)^3}{6} = \frac{64}{6} = \frac{32}{3}$ ✓.

Worked example 18: A definite integral that gives a negative value

Explain why $\int_0^3 (x^2 - 4)\,dx$ is negative and calculate the area bounded by $y = x^2 - 4$, the axes and $x = 3$ in two parts.

The curve crosses the axis at $x = 2$, so part of the region lies below it.

$\int_0^2 (x^2 - 4)dx = \left[\frac{x^3}{3} - 4x\right]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3}$: this region is below the axis.

$\int_2^3 (x^2 - 4)dx = (9 - 12) - \left(\frac{8}{3} - 8\right) = -3 + \frac{16}{3} = \frac{7}{3}$: above the axis. The whole integral $\int_0^3 = -\frac{16}{3} + \frac{7}{3} = -3$ is negative because the larger area is below. The geometric area is $\frac{16}{3} + \frac{7}{3} = \frac{23}{3} \approx 7.67$.

Note: the area under a graph in context

The area under a graph of a rate against time is the total change in the quantity. If the graph shows litres per minute, the area is in litres; if it shows metres per second, the area is in metres. Whenever you integrate in a context, write what the answer represents and its unit. Sketching the graph first helps you decide whether the region is above or below the axis, where to split the integral and whether your answer is a sensible size compared with a rough rectangle or triangle estimate.

Method: finding the area between a curve and the x-axis

1. Sketch the curve and find where it crosses the x-axis.
2. Split the interval at each crossing point.
3. Integrate each section separately.
4. Take the absolute value of any negative result and add the pieces.
5. Quote the units, such as m^2 or "square units".

Watch out: common integration mistakes

- Forgetting the constant $+c$ in an indefinite integral.
- Raising the power but dividing by the old one: $\int x^n dx = \dfrac{x^{n+1}}{n+1}$, not $\dfrac{x^{n+1}}{n}$.
- Integrating $\dfrac{1}{x^2}$ as $\ln x^2$ or x^{-1} with the wrong sign: it is $-\dfrac{1}{x}$.
- Using a single integral when the curve crosses the axis; the positive and negative areas cancel.
- Subtracting in the wrong order for the area between a curve and a line (always upper minus lower).

Note: integration as the reverse of differentiation

The fundamental theorem of calculus says that if $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$. This links the gradient of a curve with the area under another curve. Each time you integrate you can check your answer by differentiating it again: you should return to the integrand. This habit catches most errors with powers and signs, so make it automatic on every question.

Did you know?

Archimedes (287-212 BC) found the area of a parabolic segment by cutting it into infinitely many thin triangles, nearly 2,000 years before calculus. Newton and Leibniz showed that finding areas and finding gradients are opposite processes. The integral sign $\int$ is an elongated letter S, from the Latin *summa*, because an integral is a sum of infinitely many very thin strips. Engineers use integration to calculate volumes of

water behind dams and the work done by forces.

Otto's tip

Differentiate your answer to check it. If you get the original function back, you have integrated correctly.