Worksheet - Integration

Year: Year 12 · 20

Show your working.

Question 1

Question 1 – Indefinite integrals *(Section A · Skills practice)*

Find each integral. Remember the constant of integration.

a. $\int (6x^2 - 4x + 5)\,dx$ [2 marks]

b. $\int \left(3\sqrt{x} - \frac{2}{x^2}\right)dx$ [3 marks]

c. $\int (2x + 3)(x - 1)\,dx$ [3 marks]

d. $\int \frac{4x^3 - x}{x}\,dx$ [3 marks]

Question 2

Question 2 – Finding the constant *(Section A · Skills practice)*

Use the given point to find the constant of integration.

a. The curve $y = f(x)$ has $\frac{dy}{dx} = 6x - 4$ and passes through $(1, 3)$. Find y. [4 marks]

b. $f'(x) = 3x^2 - 2x$ and $f(2) = 5$. Find $f(x)$. [4 marks]

c. Find the equation of the curve with gradient $\frac{dy}{dx} = \frac{4}{\sqrt{x}}$ passing through $(4, 10)$. [4 marks]

Question 3

Question 3 – Definite integrals *(Section A · Skills practice)*

Evaluate each definite integral.

a. $\int_1^3 (2x + 1)\,dx$ [3 marks]

b. $\int_0^4 \sqrt{x}\,dx$ [3 marks]

c. $\int_1^2 \left(x^2 - \frac{1}{x^2}\right)dx$ [4 marks]

Question 4

Question 4 – Area under a curve *(Section A · Skills practice)*

Find the area of each region.

a. The region between $y = x^2 + 2$, the x-axis and the lines $x = 1$ and $x = 3$. [4 marks]
b. The region enclosed by $y = 4x - x^2$ and the x-axis. [5 marks]

Question 5

Question 5 – Areas below the axis *(Section A · Skills practice)*

The curve $y = x^2 - 4x + 3$ crosses the x-axis at $x = 1$ and $x = 3$.

a. Find $\int_1^3 (x^2 - 4x + 3)\,dx$ and explain the sign. [3 marks]
b. Find the area enclosed between the curve and the x-axis. [1 mark]
c. Find the total area between the curve, the x-axis, $x = 0$ and $x = 3$. [4 marks]

Question 6

Question 6 – Area between a curve and a line *(Section A · Skills practice)*

The curve $y = x^2$ and the line $y = 2x$ meet at two points.

a. Find the coordinates of the points of intersection. [2 marks]
b. Find the area of the region enclosed between the curve and the line. [4 marks]

Question 7

Question 7 – Filling a tank *(Section B · Real-life practice)*

Water flows into a tank at a rate of $\frac{dV}{dt} = 20 - 2t$ litres per minute for $0 \le t \le 10$, where t is in minutes. The tank initially holds 5 litres.

a. Show that $V = 5 + 20t - t^2$. [2 marks]
b. How much water is in the tank after 10 minutes? [2 marks]
c. How much water enters between $t = 2$ and $t = 6$? [3 marks]
d. The tank holds 120 litres when full. Will it overflow during these 10 minutes? [2 marks]

Question 8

Question 8 – Distance and displacement *(Section B · Real-life practice)*

A particle moves on a straight line with velocity $v = 3t^2 - 12t + 9$ m/s for $0 \le t \le 4$. Its

displacement from the origin is s metres, with $s = 0$ when $t = 0$.

a. Show that $s = t^3 - 6t^2 + 9t$. [2 marks]
b. Find the times when the particle is at rest. [2 marks]
c. Find the displacement at $t = 1$, $t = 3$ and $t = 4$. [3 marks]
d. Find the total distance travelled in the first 4 seconds. [3 marks]

Question 9

Question 9 – Soil for a flower bed *(Section B · Real-life practice)*

A garden designer plots a flower bed on a plan with units in metres. The bed lies between the x-axis and the curve $y = 6x - x^2$ for $0 \le x \le 6$.

a. Find the area of the bed. [3 marks]
b. The bed is to be filled to a depth of 0.3 m with soil. Find the volume of soil. [1 mark]
c. Soil costs £42 per cubic metre. Find the cost. [1 mark]
d. Find the greatest width of the bed (in the y-direction). [2 marks]

Question 10

Question 10 – The cost of making more items *(Section B · Real-life practice)*

The marginal cost (cost of one extra item) of producing x items is $\dfrac{dC}{dx} = 0.06x + 4$ pounds. There are fixed costs of £500.

a. Show that $C = 0.03x^2 + 4x + 500$. [2 marks]
b. Find the cost of producing 100 items. [1 mark]
c. Find the extra cost of increasing production from 100 to 150 items. [3 marks]
d. Find the average cost per item when 150 items are made. [2 marks]

Question 11

Question 11 – The arch window *(Section B · Real-life practice)*

An arched window has a straight bottom edge on the x-axis from $(0, 0)$ to $(8, 0)$ and a top edge following $y = 4x - \frac{1}{2}x^2$ (units: feet).

a. Find the height of the window at its centre. [2 marks]
b. Find the area of the window. [4 marks]
c. Glass costs £85 per square foot. Find the cost to the nearest £10. [2 marks]
d. Find the area of the rectangle enclosing the window and the fraction filled by the window. [3 marks]

Question 12

Question 12 – A growing town *(Section B · Real-life practice)*

The rate of growth of a town\'s population is $\frac{dP}{dt} = 100 + 12t$ people per year, where t is the number of years since 2020. The population in 2020 was 5 000.

a. Show that $P = 5000 + 100t + 6t^2$. [2 marks]
b. Find the population in 2030. [1 mark]
c. How many people are added between 2025 and 2030? [3 marks]
d. In which year does the population first exceed 7 000? [4 marks]

Question 13

Question 13 – Land by the river *(Section B · Real-life practice)*

On a map (units of 100 m) a road follows $y = x$ and a river follows $y = 4x - x^2$. They meet at the origin and at another point.

a. Find the other point of intersection. [3 marks]
b. Find the area between the road and the river. [4 marks]
c. Convert to hectares (1 unit^2 = 10 000 m^2 = 1 hectare). [1 mark]
d. Land costs £28 000 per hectare. Find the price. [1 mark]

Question 14

Question 14 – Average temperature *(Section B · Real-life practice)*

The temperature in a greenhouse (°C) is modelled by $T = -0.5t^2 + 6t + 8$ for $0 \le t \le 12$, where t is the number of hours after 6 am.

a. Find the maximum temperature and when it occurs. [3 marks]
b. The mean value of T over the interval is $\frac{1}{12}\int_0^{12} T\,dt$. Find it. [4 marks]
c. At what times is the temperature exactly equal to the mean value? [4 marks]

Question 15

Question 15 – Estimating an area *(Section C · Problem solving)*

The curve $y = x(4 - x)$ is shown by the table of values.

x	0	1	2	3	4
y	0	3	4	3	0

a. Check two entries of the table. [2 marks]
b. Estimate the area under the curve between $x = 0$ and $x = 4$ by joining the points with straight lines (trapezia of width 1). [3 marks]

c. Find the exact area by integration. [3 marks]
d. Find the percentage error of the estimate, and explain why it is an underestimate. [3 marks]

Question 16

Question 16 – Curve from gradient information *(Section C · Problem solving)*

A curve has gradient $\frac{dy}{dx} = ax + b$, where a and b are constants. It has a stationary point at $x = 2$ and passes through $(0, 1)$ and $(2, -3)$.

a. Show that $y = \frac{1}{2}ax^2 + bx + 1$. [2 marks]
b. Find a and b and hence y. [5 marks]
c. The curve crosses the x-axis at $x = 2 \pm \sqrt{3}$. Show that the area enclosed with the x-axis is $4\sqrt{3}$. [5 marks]

Question 17

Question 17 – Area between two curves *(Section C · Problem solving)*

The curves C_1: $y = 6 - x^2$ and C_2: $y = x^2 - 2$ enclose a region.

x	-2	-1	0	1	2
C_1	2	5	6	5	2
C_2	2	-1	-2	-1	2

a. Show that the curves meet where $x = \pm 2$. [2 marks]
b. Find the area of the enclosed region. [5 marks]
c. On a plan the units are 10 m. Convert the area to square metres. [1 mark]
d. A pond liner costs £6.50 per square metre. Find the cost to the nearest £100. [2 marks]

Question 18

Question 18 – Unknown limit of integration *(Section C · Problem solving)*

Find the value of the unknown in each part.

a. $\int_1^k (2x + 3)\,dx = 18$ with $k > 1$. Give k to 2 decimal places. [5 marks]
b. $\int_0^p (6 - 2x)\,dx = 8$. Find the possible values of p and explain geometrically why there are two. [6 marks]

Question 19

Question 19 – Total distance from a velocity table *(Section C · Problem solving)*

A cyclist\'s velocity along a straight road is $v = t^2 - 6t + 5$ m/s for $0 \le t \le 6$. Positive means forwards.

t (s)	0	1	3	5	6
v (m/s)	5	0	-4	0	5

a. Show that the cyclist is momentarily at rest at $t = 1$ and $t = 5$. [2 marks]
b. Find the displacement from the start at $t = 6$. [3 marks]
c. Find the total distance cycled in the 6 seconds. [5 marks]
d. How far from the start is the cyclist when furthest back? [2 marks]

Question 20

Question 20 – Challenge: halving an area *(Section D · Challenge)*

The region R is bounded by the curve $y = x(a - x)$ and the x-axis, where $a > 0$.

a. Show that the area of R is $\dfrac{a^3}{6}$ and find a when the area is 36. [4 marks]
b. With $a = 6$, the line $y = mx$ (0) cuts R into two parts. Show that the area between the curve and the line is $\dfrac{(6-m)^3}{6}$. [5 marks]
c. Find m so that the line splits R into two equal areas, giving m to 3 significant figures. [4 marks]