Answers - Integration

Year: Year 12

Answer to Question 1

a. Integrate term by term using $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$: increase the power by 1 and divide by the new power. $2x^3 - 2x^2 + 5x + c$. Check by differentiating: $6x^2 - 4x + 5$ ✓.

b. First write each term as a power of x: $\sqrt{x} = x^{1/2}$ and $\frac{2}{x^2} = 2x^{-2}$.
$\int (3x^{1/2} - 2x^{-2})dx = 3 \times \frac{x^{3/2}}{3/2} - 2 \times \frac{x^{-1}}{-1} + c = 2x^{3/2} + \frac{2}{x} + c$. Dividing by $\frac{3}{2}$ is the same as multiplying by $\frac{2}{3}$.

c. A product cannot be integrated term by term, so expand the brackets first.
Expand: $2x^2 + x - 3$. Integral $= \frac{2}{3}x^3 + \frac{1}{2}x^2 - 3x + c$. Check: differentiating gives $2x^2 + x - 3$ ✓.

d. Divide each term of the numerator by x before integrating. Divide first: $4x^2 - 1$.
Integral $= \frac{4}{3}x^3 - x + c$. Note that this is valid for $x \neq 0$.

Answer to Question 2

a. Integrate the gradient function, then substitute the point to find c.
$y = 3x^2 - 4x + c$. At $(1, 3)$: $3 - 4 + c = 3$, so $c = 4$. $y = 3x^2 - 4x + 4$. Check: $3 - 4 + 4 = 3$ ✓.

b. Integrate $f'(x)$, then use $f(2) = 5$. $f(x) = x^3 - x^2 + c$. $f(2) = 8 - 4 + c = 5$ gives $c = 1$. $f(x) = x^3 - x^2 + 1$. Check: $f(2) = 8 - 4 + 1 = 5$ ✓.

c. Write $\frac{4}{\sqrt{x}} = 4x^{-1/2}$; integrating $x^{-1/2}$ gives $\frac{x^{1/2}}{1/2} = 2x^{1/2}$.
$y = 4\int x^{-1/2}dx = 8x^{1/2} + c$. At $(4, 10)$: $16 + c = 10$, $c = -6$. $y = 8\sqrt{x} - 6$. Check: at $x = 4$, $8 \times 2 - 6 = 10$ ✓.

Answer to Question 3

a. Integrate, then substitute the upper limit and subtract the value at the lower limit; no constant is needed. $[x^2 + x]_1^3 = (9 + 3) - (1 + 1) = \mathbf{10}$. So $\int_1^3 (2x + 1)\,dx = 10$.

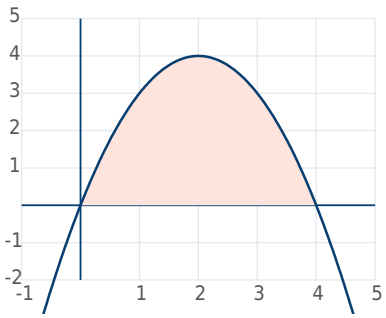
b. Write $\sqrt{x} = x^{1/2}$, which integrates to $\frac{2}{3}x^{3/2}$; note that $4^{3/2} = (\sqrt{4})^3 = 8$. $\left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3} \times 8 - 0 = \frac{16}{3}$. So the integral equals $\frac{16}{3} \approx 5.33$.

c. Write $\frac{1}{x^2} = x^{-2}$, which integrates to $-x^{-1}$; so $-\frac{1}{x^2}$ integrates to $+\frac{1}{x}$. $\left[\frac{x^3}{3} + \frac{1}{x}\right]_1^2 = \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) = \frac{19}{6} - \frac{4}{3} = \frac{11}{6}$. So the integral equals $\frac{11}{6}$.

Answer to Question 4

a. The curve $y = x^2 + 2$ is always above the x-axis, so the area is the definite integral from 1 to 3. Area $= \int_1^3 (x^2 + 2)dx = \left[\frac{x^3}{3} + 2x\right]_1^3 = (9 + 6) - \left(\frac{1}{3} + 2\right) = \frac{38}{3}$ (12.67). The area is $\frac{38}{3}$ square units.

b. First find where the curve meets the x-axis to get the limits of integration. The curve meets the axis where $x(4 - x) = 0$: $x = 0,\ 4$. Area $= \int_0^4 (4x - x^2)dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$. The curve is above the axis between the roots, so the integral is positive and the area is $\frac{32}{3}$ square units.



Answer to Question 5

a. Integrate between the two roots. $\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 = (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right) = 0 - \frac{4}{3} = -\frac{4}{3}$. It is negative because the curve is **below** the axis between 1 and 3.

b. Area $= \frac{4}{3}$ (take the modulus). An area is always positive, even when the integral is negative.

c. Split the interval at the root $x = 1$, because the curve changes sign there. From 0

to 1 the curve is above the axis: $\int_0^1 = \frac{1}{3} - 2 + 3 = \frac{4}{3}$. The total area is $\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$. Do not use $\int_0^3$ because that gives 0 (the positive and negative parts cancel).

Answer to Question 6

a. At the intersection points the two y-values are equal. $x^2 = 2x \Rightarrow x(x-2) = 0$: points $(0, 0)$ and $(2, 4)$. These are the limits of integration for part (b).
b. Integrate (upper function − lower function) between the intersection points. The line is above the curve between 0 and 2. Area $= \int_0^2 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$. The enclosed area is $\frac{4}{3}$ square units.

Answer to Question 7

a. Integrate the rate of flow to get the volume, then use the initial volume to find c. $V = \int (20 - 2t)dt = 20t - t^2 + c$; $V(0) = 5$, so $c = 5$. This is the required formula.
b. Substitute $t = 10$. $V(10) = 5 + 200 - 100 =$ **105 litres**. After 10 minutes the tank holds 105 litres.
c. The amount that enters is the definite integral of the rate from 2 to 6. $\int_2^6 (20 - 2t)dt = [20t - t^2]_2^6 = (120 - 36) - (40 - 4) = 84 - 36 =$ **48 litres**. So 48 litres flow in between the 2nd and 6th minutes.
d. The flow rate falls to 0 at $t = 10$, so V is increasing for the whole period and reaches its maximum 105 L, which is less than 120 L: it **does not** overflow.

Answer to Question 8

a. Displacement is the integral of velocity. $s = \int v\,dt = t^3 - 6t^2 + 9t + c$ and $s(0) = 0$ gives $c = 0$. This is the required expression.
b. The particle is at rest when $v = 0$. $3(t-1)(t-3) = 0$: $t = 1$ s and $t = 3$ s.
c. Substitute each time into s. $s(1) = 1 - 6 + 9 = 4$ m. $s(3) = 27 - 54 + 27 = 0$. $s(4) = 64 - 96 + 36 = 4$ m. The particle returns to the origin at $t = 3$.
d. Because the particle changes direction at $t = 1$ and $t = 3$, add the distances of each stage separately. The particle goes 0 to 4 (4 m), back from 4 to 0 (4 m), then 0 to 4 (4 m). Total distance = **12 m**, though the final displacement is only 4 m.

Answer to Question 9

a. The curve is above the x-axis between its roots 0 and 6, so integrate from 0 to 6. Area $= \int_0^6 (6x - x^2)dx = \left[3x^2 - \frac{x^3}{3}\right]_0^6 = 108 - 72 =$ **36 m**^{2}. The bed has an area of 36 m^2.
b. Volume is area multiplied by depth. $36 \times 0.3 =$ **10.8 m**^{3}. 10.8 m^3 of soil is needed.
c. Multiply the volume by the price per cubic metre. $10.8 \times 42 =$ **£453.60**. The soil costs £453.60.
d. The parabola is symmetrical, so its highest point is midway between the roots. The maximum height of the curve is at $x = 3$: $y = 18 - 9 =$ **9 m**. The greatest width of the bed is 9 m.

Answer to Question 10

a. Integrate the marginal cost; the constant is the fixed cost because $C = 500$ when $x = 0$. $C = \int (0.06x + 4)dx = 0.03x^2 + 4x + c$ and $C(0) = 500$. This is the required formula.
b. Substitute $x = 100$. $C(100) = 300 + 400 + 500 =$ **£1 200**. Producing 100 items costs £1 200.
c. Find the difference $C(150) - C(100)$. $C(150) = 675 + 600 + 500 = 1775$, so the extra cost is $1775 - 1200 =$ **£575**. (Or $\int_{100}^{150} (0.06x + 4)dx = 575$.) Increasing production to 150 items costs an extra £575.
d. The average cost is the total cost divided by the number of items. $\frac{1775}{150} =$ **£11.83**. The average cost is about £11.83 per item.

Answer to Question 11

a. By symmetry the centre is at $x = 4$, halfway between 0 and 8. At $x = 4$: $16 - 8 =$ **8 ft**. The window is 8 ft high in the middle.
b. The area is the integral of the top edge from 0 to 8. $\int_0^8 \left(4x - \frac{1}{2}x^2\right)dx = \left[2x^2 - \frac{x^3}{6}\right]_0^8 = 128 - \frac{256}{3} = \frac{128}{3} = 42.67$ ft^2. The window has an area of about 42.7 square feet.
c. Multiply the area by the price per square foot. $42.67 \times 85 = 3627$, so about **£3 630**. The glass costs about £3 630.
d. The window fits in a rectangle 8 ft wide and 8 ft high. The rectangle is $8 \times 8 = 64$ ft^2. The window fills $\frac{128/3}{64} = \frac{2}{3}$ of it. This agrees with the general fact that a parabolic arch fills two-thirds of its enclosing rectangle.

Answer to Question 12

a. Integrate the growth rate and use $P = 5000$ when $t = 0$.
$P = \int (100 + 12t)dt = 100t + 6t^2 + c$ with $P(0) = 5000$. This is the required formula.

b. 2030 corresponds to $t = 10$. $P(10) = 5000 + 1000 + 600 =$ **6 600**. The population in 2030 is 6 600.

c. The increase is the definite integral of the rate from $t = 5$ to $t = 10$.
$\int_5^{10} (100 + 12t)dt = [100t + 6t^2]_5^{10} = (1000 + 600) - (500 + 150) =$ **950**. 950 people are added between 2025 and 2030.

d. Solve $5000 + 100t + 6t^2 > 7000$ with the quadratic formula, keeping the positive root. $6t^2 + 100t - 2000 > 0$, $3t^2 + 50t - 1000 > 0$,
$t > \dfrac{-50 + \sqrt{2500 + 12000}}{6} = \dfrac{-50 + 120.4}{6} = 11.7$. So in **2032** ($t = 12$).

Answer to Question 13

a. Set the two equations equal to each other. $4x - x^2 = x \Rightarrow x(3 - x) = 0$: the other point is $(3, 3)$. The road and the river meet at the origin and at $(3, 3)$.

b. Integrate (river $-$ road) between the intersection points. The river is above the road between 0 and 3. Area $= \int_0^3 (3x - x^2)dx = \left[\frac{3}{2}x^2 - \dfrac{x^3}{3}\right]_0^3 = 13.5 - 9 =$ **4.5**.
The area is 4.5 square units.

c. Each square unit is $100 \times 100 = 10\,000$ m^2, which is 1 hectare. **4.5 hectares**.

d. Multiply the number of hectares by the price per hectare. $4.5 \times 28000 =$ **£126 000**. The land costs £126 000.

Answer to Question 14

a. Complete the square, or differentiate and set $\dfrac{dT}{dt} = 0$. $T = -0.5(t - 6)^2 + 26$: maximum **26°C** at $t = 6$ (noon).

b. Integrate from 0 to 12 and divide by the length of the interval.
$\int_0^{12} (-0.5t^2 + 6t + 8)dt = \left[-\dfrac{t^3}{6} + 3t^2 + 8t\right]_0^{12} = -288 + 432 + 96 = 240$. Mean $= \dfrac{240}{12} =$ **20°C**. The mean temperature is 20°C.

c. Set $T = 20$ and solve the quadratic using the formula.
$-0.5t^2 + 6t + 8 = 20 \Rightarrow t^2 - 12t + 24 = 0 \Rightarrow t = 6 \pm \sqrt{12} = 6 \pm 3.46$. So $t = 2.54$ and 9.46, i.e. about **8:32 am** and **3:28 pm**.

Answer to Question 15

a. Substitute into $y = x(4 - x)$. $x = 1$: $1 \times 3 = 3$ ✓. $x = 3$: $3 \times 1 = 3$ ✓. The table values are correct.

b. Each trapezium has width 1, so the area is $h\left[\frac{1}{2}(y_0 + y_4) + y_1 + y_2 + y_3\right]$. Area $\approx 1 \times \left(\frac{0+0}{2} + 3 + 4 + 3\right) = \mathbf{10}$. The estimate of the area is 10 square units.

c. Integrate $4x - x^2$ from 0 to 4. $\int_0^4 (4x - x^2)dx = \dfrac{32}{3} = 10.67$. The exact area is $\dfrac{32}{3}$ square units.

d. Percentage error $= \dfrac{\text{exact} - \text{estimate}}{\text{exact}} \times 100\%$. Error $= \dfrac{10.667 - 10}{10.667} = \mathbf{6.25\%}$. The curve is concave (it bends downwards), so each chord lies below the curve and the trapezia fill less than the full area.

Answer to Question 16

a. Integrate the gradient and use the point $(0, 1)$ to find the constant. $y = \int (ax + b)dx = \frac{1}{2}ax^2 + bx + c$ and $y(0) = 1$ gives $c = 1$. This is the required form.

b. At a stationary point the gradient is 0; the point $(2, -3)$ gives a second equation. Stationary at $x = 2$: $2a + b = 0$. Passes through $(2, -3)$: $2a + 2b + 1 = -3$, so $a + b = -2$. Subtract: $a = 2$, $b = -4$. $y = x^2 - 4x + 1$. So $a = 2$, $b = -4$ and $y = x^2 - 4x + 1$.

c. Area . Use $y = (x - 2)^2 - 3$ and let $u = x - 2$:

$\int_{-\sqrt{3}}^{\sqrt{3}} (u^2 - 3)du = \left[\dfrac{u^3}{3} - 3u\right]_{-\sqrt{3}}^{\sqrt{3}} = 2(\sqrt{3} - 3\sqrt{3}) = -4\sqrt{3}$. The area is $4\sqrt{3}$ (6.93).

Answer to Question 17

a. Set the two equations equal. $6 - x^2 = x^2 - 2 \Rightarrow x^2 = 4$, $x = \pm 2$ (as in the table).

b. Integrate $C_1 - C_2 = (6 - x^2) - (x^2 - 2) = 8 - 2x^2$ between the intersection points. C_1 is above C_2 between -2 and 2. Area $= \int_{-2}^{2} (8 - 2x^2)dx = \left[8x - \dfrac{2x^3}{3}\right]_{-2}^{2} = 2\left(16 - \dfrac{16}{3}\right) = \dfrac{64}{3}$. The enclosed area is $\dfrac{64}{3} \approx 21.3$ square units.

c. Each square unit is $10 \times 10 = 100$ m^2. $\dfrac{64}{3} \times 100 = \mathbf{2\ 133\ m^2}$.

d. Multiply the area by the price per square metre. $2133 \times 6.50 = 13\,867$, so about **£13 900**. The pond liner costs about £13 900.

Answer to Question 18

a. Integrate, substitute the limits and solve the resulting quadratic in k.
$[x^2+3x]_1^k = k^2+3k-4 = 18$, so $k^2+3k-22=0$ and $k = \dfrac{-3+\sqrt{97}}{2} = \mathbf{3.42}$ (the negative root is rejected as $k>1$). Check: $3.42^2+3\times3.42-4 \approx 18$ ✓.

b. Integrate, substitute the limits and solve the quadratic in p.
$[6x-x^2]_0^p = 6p-p^2 = 8$, so $p^2-6p+8=0$, $(p-2)(p-4)=0$: $p=2$ **or** $p=4$. The line $y=6-2x$ is above the axis for x (area 9 up to $x=3$) and below it afterwards, so the integral is 8 once when it is rising to 9 and once more after it has fallen back, since $\int_0^4 = 9-1=8$.

Answer to Question 19

a. The cyclist is at rest when $v=0$; factorise the quadratic. $(t-1)(t-5)=0$, as the table shows. So the cyclist stops at $t=1$ and $t=5$.

b. Integrate v to get s, with $s=0$ at $t=0$. $s=\dfrac{t^3}{3}-3t^2+5t$. $s(6)=72-108+30=-6$ **m** (6 m behind the start). So after 6 s the cyclist is 6 m behind the start.

c. The velocity changes sign at $t=1$ and $t=5$, so find the distance in each stage separately and add. $s(1)=\dfrac{7}{3}$; $s(5)=\dfrac{125}{3}-75+25=-\dfrac{25}{3}$. Distances: $0\to1$: $\dfrac{7}{3}$; $1\to5$: [redacted]; $5\to6$: $-6+\dfrac{25}{3}=\dfrac{7}{3}$. Total $=\dfrac{7+32+7}{3}=\dfrac{46}{3}\approx15.3$ **m**.

d. The cyclist moves backwards from $t=1$ to $t=5$, so the furthest point back is at $t=5$. Furthest back at $t=5$: $-\dfrac{25}{3}\approx-8.33$ **m**. So the cyclist is about 8.33 m behind the start.

Answer to Question 20

a. The curve meets the axis at $x=0$ and $x=a$, so integrate between these limits.
$\int_0^a(ax-x^2)dx = \left[\dfrac{ax^2}{2}-\dfrac{x^3}{3}\right]_0^a = \dfrac{a^3}{2}-\dfrac{a^3}{3}=\dfrac{a^3}{6}$. $\dfrac{a^3}{6}=36$ gives $a^3=216$, $a=$ **6**.

b. The line meets the curve where $x(6-x)=mx$, i.e. $x=0$ and $x=6-m$. Between them the curve is above the line: area $=\int_0^{6-m}\left[(6-m)x-x^2\right]dx$, which by part (a) with a replaced by $6-m$ is $\dfrac{(6-m)^3}{6}$. This is the required expression.

c. The two parts are equal when the region between curve and line is half the area

of R. Equal areas means the region between curve and line is half of 36: $\frac{(6-m)^3}{6} = 18$, so $(6-m)^3 = 108$, $6-m = 108^{1/3} = 4.762$, so $m =$ **1.24**. So the line $y = 1.24x$ halves the region.