Lesson - Indices, surds and quadratic functions

Year: Year 12

This chapter is the algebra toolkit for the whole of A level: rational indices, exact answers with surds, and every method for quadratics (factorising, completing the square, the formula and the discriminant).

Starter activity

A square tile has area 8 cm^2. Without a calculator, what is the length of one side? Is it a whole number? Write the exact length as a surd and then decide between which two whole numbers it lies. A second square has half the side length. What is its area? Compare your answer with a friend and explain what happens to the area when a length is multiplied by 2 or halved.

Laws of indices

Definition: rational index

For $a > 0$ and whole numbers m, n ($n \neq 0$), $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \left(\sqrt[n]{a}\right)^m$. A negative index means a reciprocal: $a^{-n} = \dfrac{1}{a^n}$.

Rule: laws of indices

- $a^m \times a^n = a^{m+n}$ and $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$ and $(ab)^n = a^n b^n$
- $a^0 = 1$ and $a^{-n} = \dfrac{1}{a^n}$

Worked example 1

Evaluate $16^{-3/4}$.

Take the root first: $16^{1/4} = 2$. Then cube: $2^3 = 8$. The negative power gives the reciprocal, so $16^{-3/4} = \dfrac{1}{8}$.

Worked example 2

Solve $8^x = \dfrac{1}{4}$.

Write both sides as powers of 2: $2^{3x} = 2^{-2}$, so $3x = -2$ and $x = -\frac{2}{3}$.

Worked example 3

Express $\frac{4}{\sqrt{x}}$ and $\sqrt[3]{x^2}$ as powers of x.

$\frac{4}{\sqrt{x}} = 4x^{-1/2}$ and $\sqrt[3]{x^2} = x^{2/3}$. This conversion makes differentiation and integration much easier later.

Surds

Definition: surd

A **surd** is an irrational root left in exact form, such as $\sqrt{2}$. To **rationalise the denominator** is to remove the surd from the bottom of a fraction.

Rule: surds

For positive a, b: $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$. Also $(c + \sqrt{d})(c - \sqrt{d}) = c^2 - d$.

Watch out

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$, and $(x+3)^2 \neq x^2 + 9$ (the middle term $6x$ is missing).

Method: rationalising the denominator

1. For $\frac{a}{\sqrt{b}}$, multiply top and bottom by $\sqrt{b}$.
2. For $\frac{a}{c + \sqrt{d}}$, multiply top and bottom by the conjugate $c - \sqrt{d}$ (same terms, opposite sign).
3. Use $(c + \sqrt{d})(c - \sqrt{d}) = c^2 - d$ for the new denominator and simplify.

Worked example 4

Simplify $\sqrt{75} - \sqrt{12}$, then rationalise $\frac{5}{3 - \sqrt{2}}$.

$\sqrt{75} = 5\sqrt{3}$ and $\sqrt{12} = 2\sqrt{3}$, so the first answer is $3\sqrt{3}$. Then

$$\frac{5(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})} = \frac{15+5\sqrt{2}}{9-2} = \frac{15+5\sqrt{2}}{7}.$$

Worked example 5

Find the exact area of a rectangle with sides $(3+\sqrt{5})$ cm and $(3-\sqrt{5})$ cm.

Area $= (3+\sqrt{5})(3-\sqrt{5}) = 9-5 = 4$ cm^2. The surds cancel because the brackets are conjugates.

Quadratic functions and completing the square

Definition: completed square form

Every quadratic ax^2+bx+c can be written $a(x+p)^2+q$. The graph is a parabola with **vertex** $(-p, q)$ and line of symmetry $x=-p$. If $a>0$ the vertex is the minimum point.

Method: completing the square

1. Take out the factor a from the x^2 and x terms.
2. Halve the coefficient of x inside the bracket and square it: $x^2+bx = (x+\frac{b}{2})^2 - \frac{b^2}{4}$.
3. Multiply the subtracted number by a and add the original constant.

Worked example 6

Write $2x^2-12x+5$ in the form $a(x+p)^2+q$ and state the minimum value.

$2(x^2-6x)+5 = 2[(x-3)^2-9]+5 = 2(x-3)^2-13$. The minimum value is -13 at $x=3$.

Worked example 7

Solve $x^2+4x-1=0$ exactly.

$(x+2)^2 = 5$, so $x = -2\pm\sqrt{5}$. The formula $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ comes from completing the square on $ax^2+bx+c=0$.

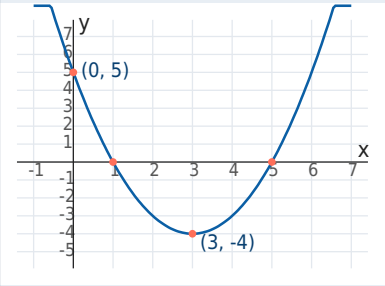
Worked example 8

Show that $x^2+6x+11>0$ for all x.

$x^2 + 6x + 11 = (x+3)^2 + 2 \geq 2$, so it is always positive.

Note: the graph

The graph of $y = x^2 - 6x + 5 = (x-3)^2 - 4$ has roots at $x = 1$ and $x = 5$, y-intercept 5 and vertex $(3, -4)$.



The discriminant

Definition: discriminant

For $ax^2 + bx + c = 0$ the **discriminant** is $\Delta = b^2 - 4ac$, the part under the root in the formula.

Key fact: roots and the discriminant

Discriminant	Roots	Graph
$b^2 - 4ac > 0$	two distinct real roots	cuts the x-axis twice
$b^2 - 4ac = 0$	one repeated root	touches the x-axis
$b^2 - 4ac$	no real roots	does not meet the x-axis

Worked example 9

Find the values of k for which $x^2 + kx + 9 = 0$ has equal roots.

We need $k^2 - 36 = 0$, so $k = \pm 6$.

Worked example 10

The line $y = 2x + k$ is a tangent to $y = x^2 + 3$. Find k.

Set equal: $x^2 - 2x + (3 - k) = 0$. A tangent touches once, so the discriminant is zero: $4 - 4(3 - k) = 0$, giving $k = 2$.

Equations quadratic in disguise

Method: substitution

1. Spot the hidden quadratic, for example $4^x = (2^x)^2$.
2. Let $y = 2^x$ and solve the quadratic in y.
3. Convert back to the original variable and check each solution.

Worked example 11

Solve $4^x - 5 \times 2^x + 4 = 0$.

With $y = 2^x$: $y^2 - 5y + 4 = 0$, so $y = 1$ or $y = 4$. Hence $2^x = 1$ or 4, so $x = 0$ or $x = 2$.

Watch out

Do not divide by x when solving (you lose the root $x = 0$). When $a \neq 1$, remember to multiply the subtracted constant by a. When asked for an *exact* answer, never give a decimal.

Worked example 12: Fractional and negative indices together

Simplify $\dfrac{(8x^6)^{2/3} \times x^{-1/2}}{2x}$ and evaluate it when $x = 4$.

$(8x^6)^{2/3} = 8^{2/3}(x^6)^{2/3} = 4x^4$. Then $\dfrac{4x^4 \times x^{-1/2}}{2x} = 2x^{4-1/2-1} = 2x^{5/2}$. When $x = 4$: $2 \times 4^{5/2} = 2 \times (\sqrt{4})^5 = 2 \times 32 = 64$. Always deal with the number part and each letter separately, and take the root first when the base is a perfect power because it keeps the numbers small.

Worked example 13: Rationalising with two surds

Write $\dfrac{3}{2\sqrt{3} - 3}$ in the form $a + b\sqrt{3}$.

Multiply top and bottom by the conjugate $2\sqrt{3} + 3$. The denominator becomes $(2\sqrt{3})^2 - 3^2 = 12 - 9 = 3$. The numerator is $3(2\sqrt{3} + 3)$. So the fraction equals $\dfrac{3(2\sqrt{3} + 3)}{3} = 3 + 2\sqrt{3}$. Check numerically: $\dfrac{3}{2 \times 1.732 - 3} = \dfrac{3}{0.4641} = 6.464$ and $3 + 2\sqrt{3} = 6.464$. The conjugate of $p + q\sqrt{r}$ is $p - q\sqrt{r}$; the product of a number and its conjugate is always rational.

Worked example 14: Completing the square when the coefficient of x^2 is not

1

Write $3x^2 - 12x + 5$ in the form $a(x+b)^2 + c$ and hence solve $3x^2 - 12x + 5 = 0$.

Take out the factor 3 from the x terms: $3(x^2 - 4x) + 5 = 3[(x-2)^2 - 4] + 5 = 3(x-2)^2 - 12 + 5 = 3(x-2)^2 - 7$. So $a = 3$, $b = -2$, $c = -7$. Setting equal to zero: $(x-2)^2 = \frac{7}{3}$, so $x = 2 \pm \sqrt{\frac{7}{3}} = 2 \pm \frac{\sqrt{21}}{3}$. The turning point is $(2, -7)$, a minimum.

Worked example 15: Using the discriminant for a tangent

Find the values of m for which the line $y = mx + 1$ is a tangent to the curve $y = x^2 + 3x + 5$.

Equate: $x^2 + 3x + 5 = mx + 1$, so $x^2 + (3-m)x + 4 = 0$. A tangent means one repeated root, so the discriminant is zero: $(3-m)^2 - 16 = 0$, giving $3 - m = \pm 4$. So $m = -1$ or $m = 7$. For $m = 7$ the equation is $x^2 - 4x + 4 = 0$ so the contact is at $x = 2$, $y = 15$, and for $m = -1$ it is $x^2 + 4x + 4 = 0$ so $x = -2$, $y = 3$.

Worked example 16: A quadratic in disguise with a root

Solve $x - 3\sqrt{x} - 4 = 0$ and explain why one of the values of $\sqrt{x}$ is rejected.

Let $u = \sqrt{x}$, so $u^2 - 3u - 4 = 0$ and $(u-4)(u+1) = 0$, giving $u = 4$ or $u = -1$. A square root (principal root) cannot be negative so $u = -1$ is rejected. Hence $\sqrt{x} = 4$ and $x = 16$. Check: $16 - 3 \times 4 - 4 = 0$. ✓.

Worked example 17: Modelling a rectangle with a surd side

A rectangle has area 12 cm^2 and one side is $(2 + \sqrt{3})$ cm. Find the other side in the form $a + b\sqrt{3}$ and the perimeter.

Other side $= \frac{12}{2+\sqrt{3}} = \frac{12(2-\sqrt{3})}{4-3} = 24 - 12\sqrt{3}$ cm ≈ 3.22 cm. Perimeter $= 2[(2+\sqrt{3}) + (24 - 12\sqrt{3})] = 2(26 - 11\sqrt{3}) = 52 - 22\sqrt{3} \approx 13.9$ cm. Always rationalise the denominator before adding.

Worked example 18: Range of values with the discriminant

Find the values of k for which $x^2 + kx + k + 3 > 0$ for all real x.

The curve $y = x^2 + kx + k + 3$ is a U-shape, so it lies completely above the x-axis exactly when it has no real roots. That needs a negative discriminant: $k^2 - 4(k+3)$, so $k^2 - 4k - 12$ and $(k-6)(k+2)$. A U-shaped quadratic is negative between its roots, so -2. Test: $k = 0$ gives $x^2 + 3 > 0$ for all x, which is true, while $k = 7$ gives $x^2 + 7x + 10 = (x+2)(x+5)$, which is negative for -5.

Worked example 19: Solving a quadratic inequality

Solve $2x^2 - x - 10$ and write the solution set in inequality notation.

First solve the equation: $2x^2 - x - 10 = (2x - 5)(x + 2) = 0$, so $x = \frac{5}{2}$ or $x = -2$. Sketch a U-shaped parabola crossing the x-axis at -2 and 2.5. The curve is below the axis (negative) between the roots, so the solution is -2. Check with $x = 0$: -10 is true, which lies inside the interval; and $x = 3$ gives $18 - 3 - 10 = 5 > 0$, outside the interval, as expected.

Worked example 20: Surds in an area problem

A square has sides of length $(3 + 2\sqrt{2})$ cm. Show that its area is $(17 + 12\sqrt{2})$ cm^2 and find the exact length of its diagonal.

Area $= (3 + 2\sqrt{2})^2 = 9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2}$ cm^2. The diagonal of a square with side s is $s\sqrt{2}$, so the diagonal is $(3 + 2\sqrt{2})\sqrt{2} = 3\sqrt{2} + 4$ cm. Numerically the side is 5.83 cm, the area is 33.97 cm^2 and the diagonal is 8.24 cm; confirm with Pythagoras, $\sqrt{2 \times 33.97} = 8.24$.

Note: reading a quadratic model

In a model such as $h = -5(t - 2)^2 + 21.5$ the numbers have meanings. The bracket tells you when the extreme value occurs (here after 2 seconds); the constant at the end gives the extreme value itself (21.5 m); and the coefficient in front tells you how sharply the curve bends. Whenever a question asks for the "greatest" or "least" value, completing the square is the quickest route because it gives both pieces of information at once. Always give units and interpret your answer in words.

Method: sketching a quadratic from its completed-square form

1. Write the function as $a(x + p)^2 + q$. The vertex is $(-p, q)$.
2. If $a > 0$ the curve is a U-shape (minimum); if a it is an N-shape (maximum).
3. Find the y-intercept by putting $x = 0$.
4. Use the discriminant to decide whether the curve crosses the x-axis: two, one (touching) or none.
5. Label the vertex, the intercepts and the line of symmetry $x = -p$.

Watch out: common errors with indices, surds and quadratics

- Writing $(a + b)^2 = a^2 + b^2$ or $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$. Both are false: $(3 + 4)^2 = 49$ not 25.
- Forgetting that a^{-n} means a reciprocal, not a negative number: $2^{-3} = \frac{1}{8}$, not -8.
- Multiplying by $\sqrt{a}$ when rationalising a denominator of the form $p + \sqrt{a}$. You

must use the conjugate $p - \sqrt{a}$.
- In completed-square form, taking the sign of p wrongly: $x^2 + 6x = (x + 3)^2 - 9$, so the vertex x-coordinate is -3.
- Dividing both sides of $x^2 = 3x$ by x, which loses the solution $x = 0$.

Did you know?

The ancient Babylonians solved quadratic equations over 3,500 years ago using the method that we now call completing the square. The Greek mathematician Pythagoras and his followers were horrified to discover numbers such as $\sqrt{2}$ that cannot be written as a fraction: according to legend, the discoverer was thrown overboard. Today, completing the square is used to design satellite dishes, the arches of bridges and the path of a basketball.

Otto's tip

The discriminant is the part under the square root. If it is negative you cannot take the root, so there are no real solutions. Write the condition first ("equal roots: $b^2 - 4ac = 0$ ") to bag the method mark.