Worksheet - Indices, surds and quadratic functions

Year: Year 12 · 20

Show your working.

Question 1

Question 1 – Laws of indices *(Section A · Skills practice)*

Work out each value without a calculator, then simplify the algebraic expression.

a. $27^{2/3}$ [2 marks]

b. $16^{-3/4}$ [2 marks]

c. $\left(\frac{8}{27}\right)^{-2/3}$ [2 marks]

d. $\frac{(2x^3)^2 \times 3x^{-5}}{6x^{-1}}$ [3 marks]

Question 2

Question 2 – Simplifying surds *(Section A · Skills practice)*

Write each answer in the form $a + b\sqrt{c}$ or $a\sqrt{c}$ where a, b, c are integers.

a. $\sqrt{75} + \sqrt{12} - \sqrt{27}$ [3 marks]

b. $(3 + \sqrt{5})(2 - \sqrt{5})$ [2 marks]

c. $\frac{6}{\sqrt{3}}$ [2 marks]

d. $\frac{4}{3 - \sqrt{5}}$ [3 marks]

e. $\frac{\sqrt{2} + 1}{\sqrt{2} - 1}$ [3 marks]

Question 3

Question 3 – Completing the square *(Section A · Skills practice)*

Write each function in completed-square form, state the coordinates of the turning point, and say whether it is a maximum or minimum.

a. $f(x) = x^2 - 8x + 19$ [3 marks]

b. $g(x) = -2x^2 + 12x - 7$ [4 marks]

c. Solve $g(x) = 0$, giving exact answers. [3 marks]

Question 4

Question 4 – Solving quadratic equations *(Section A · Skills practice)*

Solve each equation. Give exact answers where they are not integers or fractions.

a. $2x^2 - 5x - 12 = 0$ [3 marks]
b. $x^2 + 6x - 2 = 0$ [3 marks]
c. $3x^2 - 2x - 4 = 0$ [3 marks]

Question 5

Question 5 – The discriminant *(Section A · Skills practice)*

Use the discriminant $b^2 - 4ac$ to answer each part.

a. The equation $x^2 + kx + 9 = 0$ has equal roots. Find the possible values of k. [3 marks]
b. The equation $2x^2 - 4x + k = 0$ has no real roots. Find the range of values of k. [3 marks]
c. The equation $kx^2 + 8x + 2 = 0$ has two real roots. Find the range of values of k. [4 marks]

Question 6

Question 6 – Quadratics in disguise *(Section A · Skills practice)*

Use a substitution to solve each equation.

a. $x^4 - 13x^2 + 36 = 0$ [3 marks]
b. $x - 5\sqrt{x} + 6 = 0$ [3 marks]
c. $2^{2x} - 6 \times 2^x + 8 = 0$ [4 marks]

Question 7

Question 7 – Paving the patio *(Section B · Real-life practice)*

A rectangular patio has length $(5 + \sqrt{2})$ m and width $(5 - \sqrt{2})$ m. Paving slabs cost £18.90 per square metre and edging costs £14.50 per metre.

a. Show that the area of the patio is exactly 23 m^2. [2 marks]
b. Find the exact perimeter. [2 marks]
c. Show that the length of a diagonal is $3\sqrt{6}$ m. [3 marks]
d. Work out the total cost of the paving and edging. [3 marks]

Question 8

Question 8 – The football *(Section B · Real-life practice)*

A football is kicked from the ground. Its height h metres after t seconds is modelled by $h = 1.5 + 20t - 5t^2$, where 1.5 m is the height at which it is struck.

a. Write h in the form $a - b(t - c)^2$. [3 marks]
b. State the maximum height and the time at which it occurs. [2 marks]
c. The ball hits the ground when $h = 0$. Find this time to 2 decimal places. [3 marks]
d. For how long is the ball higher than 15 m? Give your answer to 2 decimal places. [3 marks]

Question 9

Question 9 – Ticket prices and profit *(Section B · Real-life practice)*

A school sells tickets for its summer show at £x each. Past sales suggest that $200 - 10x$ tickets are sold. Each ticket costs the school £3 to produce (programme and refreshments).

a. Show that the profit is $P = -10x^2 + 230x - 600$ pounds. [2 marks]
b. Find the ticket price that gives the greatest profit and state this profit. [4 marks]
c. Find the prices at which the school breaks even. [2 marks]
d. The school wants a profit of at least £500. Find the range of prices, to the nearest penny. [4 marks]

Question 10

Question 10 – Triangular sail *(Section B · Real-life practice)*

A sailing club makes a triangular sail in the shape of an isosceles triangle with base $4\sqrt{3}$ m and perpendicular height 6 m. The cloth costs £12.40 per square metre.

a. Show that the area of the sail is $12\sqrt{3}$ m^2. [2 marks]
b. Find the exact length of each sloping side. Write your answer in the form $a\sqrt{b}$. [3 marks]
c. Work out the cost of the cloth to the nearest pound. [3 marks]
d. The club wants a similar sail with area $48\sqrt{3}$ m^2. Show that all lengths must be doubled. [2 marks]

Question 11

Question 11 – Fencing a pen *(Section B · Real-life practice)*

A farmer has 60 m of fencing to make a rectangular sheep pen against a straight wall. The wall forms one side, so only three sides need fencing. The width of the pen (the sides at right angles to the wall) is x metres.

a. Show that the area of the pen is $A = 60x - 2x^2$ m^2. [2 marks]

b. Complete the square to find the maximum area and the dimensions that give it. [4 marks]
c. The farmer needs an area of at least 400 m^2. Find the range of possible widths. [3 marks]
d. Explain why the farmer cannot enclose 500 m^2 using this fencing. [2 marks]

Question 12

Question 12 – Planet orbits *(Section B · Real-life practice)*

Kepler found that the orbital period T (in Earth years) of a planet is related to its distance r from the Sun (in astronomical units, AU) by $T = r^{3/2}$.

Planet	Distance r (AU)
Mars	1.52
Jupiter	5.20
Saturn	9.54

a. Find T for Mars and Jupiter, to 2 decimal places. [3 marks]
b. A comet has period 64 years. Find its mean distance from the Sun. [3 marks]
c. Saturn\'s period is 29.5 years. Use the model to find r and compare with the table. [3 marks]
d. Show that $T^2 = r^3$. [1 mark]

Question 13

Question 13 – A cycle path and a canal *(Section B · Real-life practice)*

On a map (units: hundreds of metres) a canal bank follows the curve $y = x^2 - 4x + 7$. A straight cycle path is the line $y = 2x + c$, where c depends on where it is built.

a. Show that the x-coordinates of the points where the path meets the canal satisfy $x^2 - 6x + 7 - c = 0$. [2 marks]
b. Find the value of c for which the path just touches the canal bank. [3 marks]
c. Find the coordinates of the point of contact. [2 marks]
d. For what values of c does the path cross the canal at two points? [2 marks]

Question 14

Question 14 – Kiosk profit with a square root *(Section B · Real-life practice)*

A seaside kiosk opens for the season. Its weekly profit, in hundreds of pounds, is modelled by $P = t - 7\sqrt{t} + 10$, where t is the number of weeks since opening ($t > 0$).

a. Show that $P = 0$ when $t = 4$ and when $t = 25$. [3 marks]
b. For which weeks is the kiosk making a loss? [2 marks]
c. Find the greatest weekly loss, in pounds. [4 marks]

Question 15

Question 15 – Fencing a field: an investigation *(Section C · Problem solving)*

A farmer has 150 m of fencing to enclose a rectangular field with an area as large as possible. Let one side be x m. The second side is therefore $(75 - x)$ m. Here are some values of the area $A = x(75 - x)$.

x (m)	10	20	30	40	50	60
A (m^2)	650	1100	1350	1400	1250	900

a. Check two of the table values by calculation. [2 marks]
b. Write A in completed-square form and find the maximum area. [3 marks]
c. The farmer wants an area of exactly 1200 m^2. Find both possible values of x, to 1 decimal place. [4 marks]
d. Use the discriminant to show that an area of 1450 m^2 is impossible. [3 marks]

Question 16

Question 16 – Exact geometry with surds *(Section C · Problem solving)*

Triangle ABC has a right angle at B. $AB = (\sqrt{5} + 1)$ cm and $BC = (\sqrt{5} - 1)$ cm.

a. Show that the area of the triangle is exactly 2 cm^2. [2 marks]
b. Show that $AC = 2\sqrt{3}$ cm. [3 marks]
c. Show that $\tan A = \dfrac{3 - \sqrt{5}}{2}$. [4 marks]
d. Find the exact perimeter of the triangle. [2 marks]

Question 17

Question 17 – Relationships between the roots *(Section C · Problem solving)*

The quadratic equation $x^2 - 6x + k = 0$ has roots α and β.

a. Find the range of k for which the equation has two distinct real roots. [2 marks]
b. Given that one root is twice the other, find k. [4 marks]
c. Given that the roots differ by 4, find k and the roots. [4 marks]
d. Show that when 0 both roots are positive. [2 marks]

Question 18

Question 18 – Index equations *(Section C · Problem solving)*

Solve each equation, showing your method.

a. $8^{x-1} = 4^{x+2}$ [4 marks]
b. $9^x - 4 \times 3^{x+1} + 27 = 0$ [5 marks]

c. $x^{3/2} = 125$ [2 marks]

d. $\dfrac{2^{x+3}}{4^{x-1}} = 16$ [4 marks]

Question 19

Question 19 – The arch of a bridge *(Section C · Problem solving)*

A footbridge has a parabolic arch. Measured in metres from the left foot of the arch, the height of the arch is h at horizontal distance x. The arch is 20 m wide at ground level and 8 m high at its highest point, which is in the middle.

x (m)	0	5	10	15	20
h (m)	0	6	8	6	0

a. Show that $h = -0.08x^2 + 1.6x$ fits the table. [3 marks]

b. A narrowboat is 6 m wide and 6.5 m tall. It passes under the centre of the arch. Show that it fits. [3 marks]

c. Find the width of the arch at a height of 7 m. [3 marks]

d. Find the widest 7 m high boat. How does it compare with part (c)? [2 marks]

Question 20

Question 20 – Challenge: a quadratic that always has two roots *(Section D · Challenge)*

Consider the equation $x^2 + kx + (k - 2) = 0$ where k is a real constant.

a. Prove that the equation has two distinct real roots for all values of k. [4 marks]

b. Let the roots be α and β. Show that $\alpha^2 + \beta^2 = k^2 - 2k + 4$ and find its minimum value. [5 marks]

c. For what values of k are both roots negative? [3 marks]