Answers - Indices, surds and quadratic functions

Year: Year 12

Answer to Question 1

a. The denominator 3 of the power means a cube root and the numerator 2 means squaring; taking the root first keeps the numbers small. $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = \mathbf{9}$. So $27^{2/3} = 9$.

b. A negative power means the reciprocal; then take the fourth root and cube it. $16^{-3/4} = \dfrac{1}{16^{3/4}} = \dfrac{1}{(\sqrt[4]{16})^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$. So $16^{-3/4} = \frac{1}{8}$.

c. A negative power inverts the fraction: $\left(\dfrac{27}{8}\right)^{2/3} = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$. Check: $\left(\frac{9}{4}\right)^{-1} = \frac{4}{9}$ and $\left(\frac{8}{27}\right)^{2/3} = \frac{4}{9}$ ✓.

d. Use $(ab)^n = a^n b^n$, $x^m \times x^n = x^{m+n}$ and $x^m \div x^n = x^{m-n}$. Numerator: $(2x^3)^2 \times 3x^{-5} = 4x^6 \times 3x^{-5} = 12x$. Then $\dfrac{12x}{6x^{-1}} = 2x^{1-(-1)} = 2x^2$. So the expression simplifies to $2x^2$.

Answer to Question 2

a. Simplify each surd by taking out the largest square factor: $\sqrt{75} = \sqrt{25 \times 3}$, $\sqrt{12} = \sqrt{4 \times 3}$, $\sqrt{27} = \sqrt{9 \times 3}$. $5\sqrt{3} + 2\sqrt{3} - 3\sqrt{3} = 4\sqrt{3}$. Like surds can be collected just like like terms.

b. Expand the brackets term by term, using $\sqrt{5} \times \sqrt{5} = 5$. $6 - 3\sqrt{5} + 2\sqrt{5} - 5 = 1 - \sqrt{5}$. This is in the form $a + b\sqrt{c}$ with $a = 1$, $b = -1$, $c = 5$.

c. To rationalise the denominator, multiply by $\dfrac{\sqrt{3}}{\sqrt{3}}$, which equals 1. Multiply top and bottom by $\sqrt{3}$: $\dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$. The denominator is now an integer.

d. For a denominator $a - \sqrt{b}$, multiply top and bottom by the conjugate $a + \sqrt{b}$; the denominator becomes $a^2 - b$. Multiply by the conjugate $3 + \sqrt{5}$: $\dfrac{4(3+\sqrt{5})}{9-5} = \dfrac{4(3+\sqrt{5})}{4} = 3 + \sqrt{5}$. So $\dfrac{4}{3-\sqrt{5}} = 3 + \sqrt{5}$.

e. Multiply top and bottom by the conjugate of the denominator. Multiply by $\sqrt{2} + 1$:

$\frac{(\sqrt{2}+1)^2}{2-1} = 2+2\sqrt{2}+1 = 3+2\sqrt{2}$. Check: $\frac{2.414}{0.414} \approx 5.83$ and $3+2\sqrt{2} \approx 5.83$ ✓.

Answer to Question 3

a. Halve the coefficient of x to get $(x-4)^2$, then subtract $4^2 = 16$ to compensate. $x^2 - 8x + 19 = (x-4)^2 - 16 + 19 = (x-4)^2 + 3$. Turning point $(4, 3)$, a **minimum** (positive x^2 coefficient).

b. Take out the factor -2 from the terms in x first, then complete the square inside the bracket. $-2(x^2 - 6x) - 7 = -2[(x-3)^2 - 9] - 7 = -2(x-3)^2 + 18 - 7 = -2(x-3)^2 + 11$. Turning point $(3, 11)$, a **maximum**. A negative coefficient of x^2 gives a maximum.

c. Use the completed-square form and rearrange to isolate $(x-3)^2$. $-2(x-3)^2 + 11 = 0$ gives $(x-3)^2 = \frac{11}{2}$, so $x = 3 \pm \sqrt{\frac{11}{2}} = 3 \pm \frac{\sqrt{22}}{2}$. (Check: $\sqrt{5.5} \approx 2.345$.) So the two exact roots are $x = 3 \pm \frac{\sqrt{22}}{2}$, approximately 0.655 and 5.35.

Answer to Question 4

a. Look for two numbers with product $2 \times (-12) = -24$ and sum -5: these are -8 and 3. Factorise: $(2x+3)(x-4) = 0$, so $x = -\frac{3}{2}$ or **4**. Check: $2(16) - 20 - 12 = 0$ ✓.

b. The expression does not factorise, so complete the square. Complete the square: $(x+3)^2 - 9 - 2 = 0$, $(x+3)^2 = 11$, $x = -3 \pm \sqrt{11}$. These are the exact roots.

c. Use the quadratic formula with $a = 3$, $b = -2$, $c = -4$, and simplify $\sqrt{52} = 2\sqrt{13}$. Formula: $x = \frac{2 \pm \sqrt{4+48}}{6} = \frac{2 \pm \sqrt{52}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{1 \pm \sqrt{13}}{3}$. These are the exact roots, approximately 1.54 and -0.869.

Answer to Question 5

a. Equal roots means the discriminant is zero; here $a = 1$, $b = k$, $c = 9$. $k^2 - 36 = 0$, so $k = \pm 6$. Check: $k = 6$ gives $(x+3)^2 = 0$ ✓.

b. No real roots means the discriminant is negative; here $a = 2$, $b = -4$, $c = k$. $16 - 8k$, so $8k > 16$ and $k > 2$. So the equation has no real roots when $k > 2$.

c. Real roots need $b^2 - 4ac \geq 0$; here $a = k$, $b = 8$, $c = 2$. $64 - 8k \geq 0$ gives $k \leq 8$.

Also $k \neq 0$, otherwise the equation is not quadratic. So $k \leq 8$**,** $k \neq 0$ (two distinct real roots need k).

Answer to Question 6

a. $x^4 = (x^2)^2$, so the equation is a quadratic in x^2. Let $y = x^2$: $y^2 - 13y + 36 = 0$, $(y-4)(y-9) = 0$, so $x^2 = 4$ or 9 and $x = \pm 2$**,** ± 3. All four values are solutions because both 4 and 9 are positive.

b. $x = (\sqrt{x})^2$, so the equation is a quadratic in $\sqrt{x}$. Let $u = \sqrt{x}$: $u^2 - 5u + 6 = 0$, $(u-2)(u-3) = 0$, so $\sqrt{x} = 2$ or 3 and $x =$ **4** or **9**. Check: $4 - 10 + 6 = 0$ ✓ and $9 - 15 + 6 = 0$ ✓.

c. $2^{2x} = (2^x)^2$, so the equation is a quadratic in 2^x. Let $y = 2^x$: $y^2 - 6y + 8 = 0$, $(y-2)(y-4) = 0$, so $2^x = 2$ or 4 and $x =$ **1** or **2**. Check: $16 - 24 + 8 = 0$ ✓.

Answer to Question 7

a. Multiply length by width; the brackets form a difference of two squares, $(a+b)(a-b) = a^2 - b^2$. Area $= (5+\sqrt{2})(5-\sqrt{2}) = 25 - 2 =$ **23 m**^{2} (difference of two squares). The surds cancel, so the area is a whole number.

b. The perimeter is twice the sum of the length and the width. Perimeter $= 2[(5+\sqrt{2}) + (5-\sqrt{2})] = 2 \times 10 =$ **20 m**. The $\sqrt{2}$ terms cancel in the sum.

c. Use Pythagoras with the length and width, expanding each square with $(a \pm b)^2 = a^2 \pm 2ab + b^2$. Diagonal $^2 = (5+\sqrt{2})^2 + (5-\sqrt{2})^2 = (27 + 10\sqrt{2}) + (27 - 10\sqrt{2}) = 54$, so the diagonal is $\sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$ m. This is the required result.

d. Paving is charged per square metre (area) and edging per metre (perimeter). Paving: $23 \times 18.90 = £434.70$. Edging: $20 \times 14.50 = £290$. Total $= 434.70 + 290 =$ **£724.70**. The total cost is £724.70.

Answer to Question 8

a. Take out the factor -5 from the terms in t, then complete the square inside the bracket. $h = -5(t^2 - 4t) + 1.5 = -5[(t-2)^2 - 4] + 1.5 = -5(t-2)^2 + 21.5$.

b. Since $-5(t-2)^2 \leq 0$, h is largest when $t - 2 = 0$. Maximum height **21.5 m** at $t =$ **2 s**. The ball reaches 21.5 m after 2 seconds.

c. Use the quadratic formula on $5t^2 - 20t - 1.5 = 0$ and reject the negative time. $5t^2 - 20t - 1.5 = 0$. $t = \dfrac{20 \pm \sqrt{400 + 30}}{10} = \dfrac{20 \pm 20.736}{10}$. The positive root is $t =$ **4.07 s** (the other is negative). The ball hits the ground after about 4.07 s.

d. Use the completed-square form to set up the inequality $h > 15$. $-5(t-2)^2 + 21.5 > 15$ gives $(t-2)^2$, so $2 - \sqrt{1.3}$. Duration $= 2\sqrt{1.3} =$ **2.28 s**.

The ball is higher than 15 m for about 2.28 s.

Answer to Question 9

a. Profit is the profit per ticket multiplied by the number of tickets sold. Profit per ticket $= x - 3$, tickets sold $= 200 - 10x$, so $P = (x - 3)(200 - 10x) = 200x - 10x^2 - 600 + 30x = -10x^2 + 230x - 600$. This is the required expression.

b. Complete the square to find the maximum. $P = -10(x^2 - 23x) - 600 = -10(x - 11.5)^2 + 1322.5 - 600 = -10(x - 11.5)^2 + 722.5$. Greatest profit **£722.50** at a price of **£11.50** (85 tickets sold).

c. Breaking even means $P = 0$; divide by -10 and factorise. $P = 0$: $x^2 - 23x + 60 = 0$, $(x - 3)(x - 20) = 0$, so **£3** and **£20**. At these prices the school makes no profit.

d. Use the completed-square form and solve the inequality. $-10(x - 11.5)^2 + 722.5 \geq 500$ gives $(x - 11.5)^2 \leq 22.25$, so $|x - 11.5| \leq 4.717$. Lower bound 6.78, upper 16.22. So prices from about **£6.79 to £16.21** (rounded so the profit is still at least £500).

Answer to Question 10

a. Use area $= \frac{1}{2} \times \text{base} \times \text{height}$. Area $= \frac{1}{2} \times 4\sqrt{3} \times 6 = 12\sqrt{3}$ m^2. This is the required result.

b. The height bisects the base of an isosceles triangle, giving a right-angled triangle; use Pythagoras. Half the base is $2\sqrt{3}$. Side$^2 = (2\sqrt{3})^2 + 6^2 = 12 + 36 = 48$, so the side is $\sqrt{48} = 4\sqrt{3}$ **m**. (So the sail is equilateral.)

c. Find the area as a decimal, then multiply by the price. $12\sqrt{3} = 20.78$ m^2. Cost $= 20.78 \times 12.40 = 257.7 \approx$ **£258**. The cloth costs about £258.

d. Areas scale with the square of lengths. $\dfrac{48\sqrt{3}}{12\sqrt{3}} = 4 = 2^2$, so lengths double. So each length must be multiplied by 2.

Answer to Question 11

a. Two sides of length x use $2x$ metres of fencing; the rest forms the side parallel to the wall. Length parallel to the wall $= 60 - 2x$, so $A = x(60 - 2x) = 60x - 2x^2$. This is the required formula.

b. Take out the factor -2 and complete the square. $A = -2(x^2 - 30x) = -2[(x - 15)^2 - 225] = -2(x - 15)^2 + 450$. Maximum area **450 m**^{2} when $x = 15$ m and length $= 60 - 30 = 30$ m. So the pen should be 15 m wide and 30 m long.

c. Use the completed-square form and solve the inequality. $-2(x-15)^2+450 \geq 400$ gives $(x-15)^2 \leq 25$, so $10 \leq x \leq 20$. Any width from 10 m to 20 m gives at least 400 m^2.
d. $-2(x-15)^2+450=500$ would need $(x-15)^2=-25$, which is impossible. The maximum area is only 450 m^2. So 500 m^2 cannot be enclosed.

Answer to Question 12

a. Substitute each distance into $T=r^{3/2}$. Mars: $1.52^{1.5}=1.874 \approx$ **1.87 years**. Jupiter: $5.2^{1.5}=11.858 \approx$ **11.86 years**. Mars takes about 1.87 years and Jupiter about 11.86 years to orbit the Sun.
b. Raise both sides to the power $\frac{2}{3}$, the reciprocal of $\frac{3}{2}$. $r^{3/2}=64$, so $r=64^{2/3}=(\sqrt[3]{64})^2=4^2=$ **16 AU**. The comet's mean distance is 16 AU.
c. Use $r=T^{2/3}$ with $T=29.5$. $r=29.5^{2/3}=9.55$ AU, very close to the tabulated 9.54. So the model fits Saturn very well.
d. Square both sides and use $(a^m)^n=a^{mn}$. $T^2=\left(r^{3/2}\right)^2=r^3$. This is Kepler's third law.

Answer to Question 13

a. At an intersection the two y-values are equal. Equate: $x^2-4x+7=2x+c \Rightarrow x^2-6x+7-c=0$. This is the required equation.
b. The path just touches the bank when the quadratic has equal roots. Tangent means one repeated root, so the discriminant is zero: $36-4(7-c)=0$, so $8+4c=0$ and $c=-2$. So $c=-2$.
c. Solve the equation with $c=-2$. With $c=-2$: $x^2-6x+9=0$, $(x-3)^2=0$, $x=3$ and $y=2(3)-2=4$. Point $(3,4)$. The path touches the canal at $(3,4)$.
d. Two crossing points need two distinct roots. Two crossing points need discriminant >0: $8+4c>0$, so $c>-2$. So the path crosses the canal twice when $c>-2$.

Answer to Question 14

a. Since $t=(\sqrt{t})^2$, the expression is a quadratic in $\sqrt{t}$. Let $u=\sqrt{t}$: $u^2-7u+10=0$, $(u-2)(u-5)=0$, so $u=2$ or 5 and $t=4$ or 25. Check: $4-14+10=0$ ✓ and $25-35+10=0$ ✓.
b. The quadratic $u^2-7u+10$ has a positive u^2 coefficient, so it is negative between its roots. The expression is negative between its roots, so for 4 (weeks 5 to 24). The kiosk makes a loss between weeks 4 and 25.
c. Complete the square in u. In terms of u: $P=u^2-7u+10=(u-3.5)^2-2.25$. The minimum is -2.25 when $u=3.5$, i.e. $t=12.25$. The greatest loss is

$2.25 \times 100 =$ **£225**. The greatest weekly loss is £225, about 12 weeks after opening.

Answer to Question 15

a. Substitute two values of x into $A = x(75 - x)$. $x = 30$: $30 \times 45 = 1350$. $x = 60$: $60 \times 15 = 900$. Both match the table.

b. Expand and complete the square. $A = -(x^2 - 75x) = -(x - 37.5)^2 + 1406.25$. The maximum area is **1406.25 m^2**, achieved by a square of side 37.5 m.

c. Set up the quadratic and use the formula. $x(75 - x) = 1200 \Rightarrow x^2 - 75x + 1200 = 0$. Discriminant $= 5625 - 4800 = 825$. $x = \dfrac{75 \pm \sqrt{825}}{2} = \dfrac{75 \pm 28.72}{2}$, so $x =$ **23.1** or **51.9**. (The two fields are the same rectangle turned round.) The field could be 23.1 m by 51.9 m.

d. Write the equation $x(75 - x) = 1450$ and compute its discriminant. $x^2 - 75x + 1450 = 0$ has discriminant $5625 - 5800 = -175$, so no real solution: 1450 exceeds the maximum 1406.25. So an area of 1450 m^2 is impossible.

Answer to Question 16

a. The two shorter sides meet at the right angle, so use $\frac{1}{2} \times AB \times BC$ and the difference of two squares. Area $= \frac{1}{2}(\sqrt{5} + 1)(\sqrt{5} - 1) = \frac{1}{2}(5 - 1) = 2$ cm^2. The surds cancel, so the area is exactly 2 cm^2.

b. Use Pythagoras: $AC^2 = AB^2 + BC^2$, expanding each square carefully. $AC^2 = (\sqrt{5} + 1)^2 + (\sqrt{5} - 1)^2 = (6 + 2\sqrt{5}) + (6 - 2\sqrt{5}) = 12$, so $AC = \sqrt{12} = 2\sqrt{3}$. So $AC = 2\sqrt{3}$ cm, as required.

c. $\tan A$ is the side opposite A (that is BC) divided by the side adjacent to A (that is AB); then rationalise the denominator. $\tan A = \dfrac{BC}{AB} = \dfrac{\sqrt{5} - 1}{\sqrt{5} + 1}$. Multiply by $\dfrac{\sqrt{5} - 1}{\sqrt{5} - 1}$: $\dfrac{(\sqrt{5} - 1)^2}{5 - 1} = \dfrac{6 - 2\sqrt{5}}{4} = \dfrac{3 - \sqrt{5}}{2}$. This is the required result.

d. Add the three side lengths; the $+1$ and -1 cancel. Perimeter $= (\sqrt{5} + 1) + (\sqrt{5} - 1) + 2\sqrt{3} = 2\sqrt{5} + 2\sqrt{3}$ **cm**. The perimeter is $(2\sqrt{5} + 2\sqrt{3})$ cm, about 7.94 cm.

Answer to Question 17

a. Two distinct real roots need $b^2 - 4ac > 0$. $36 - 4k > 0$, so k.

b. For $x^2 + bx + c = 0$, the sum of the roots is $-b$ and the product is c. Let the roots be α and 2α. Sum: $3\alpha = 6$, $\alpha = 2$. Product: $2\alpha^2 = 8 = k$. So $k = 8$ (roots 2 and 4).

Check: $2+4=6$ ✓ and $2\times 4=8$ ✓.

c. $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=36-4k=16$, so $k=5$. The equation $x^2-6x+5=0$ has roots **1 and 5**. Check: $5-1=4$ ✓.

d. Sum of roots $=6>0$ and product $=k>0$. A positive product means the roots have the same sign, and a positive sum means that sign is positive. Real roots need $k\leq 9$. So for 0 both roots are positive.

Answer to Question 18

a. 8 and 4 are both powers of 2, so write both sides with base 2 and compare the indices. Write both sides as powers of 2: $2^{3(x-1)}=2^{2(x+2)}$, so $3x-3=2x+4$ and $x=$ **7**. Check: $8^6=2^{18}=4^9$ ✓.

b. Write every term in terms of 3^x. Let $y=3^x$. Then $9^x=y^2$ and $3^{x+1}=3y$. $y^2-12y+27=0$, $(y-3)(y-9)=0$, so $3^x=3$ or 9 and $x=$ **1** or **2**.

c. Raise both sides to the power $\frac{2}{3}$. $x=125^{2/3}=(\sqrt[3]{125})^2=5^2=$ **25**. Check: $25^{3/2}=5^3=125$ ✓.

d. Write 4^{x-1} as 2^{2x-2} and use the division law. $\dfrac{2^{x+3}}{2^{2x-2}}=2^{x+3-2x+2}=2^{5-x}$. So $2^{5-x}=2^4$ and $x=$ **1**. Check: $\dfrac{2^4}{4^0}=16$ ✓.

Answer to Question 19

a. Substitute each value of x from the table into the formula. $x=5$: $-0.08(25)+8=6$. $x=10$: $-8+16=8$. $x=15$: $-0.08(225)+24=6$. $x=0$ and $x=20$: 0 ($-0.08\times 400+32=0$). All match. So the model fits all the data.

b. The boat is centred at $x=10$, so it occupies 3 m on each side; the lowest point of the arch above the boat is at its edges. The boat occupies $7\leq x\leq 13$. The arch is lowest at the edges: $h(7)=-0.08\times 49+11.2=7.28>6.5$. So the boat **fits** with 0.78 m to spare.

c. Write $h=-0.08(x-10)^2+8$ and set $h=7$. $-0.08(x-10)^2+8=7$ gives $(x-10)^2=12.5$, so $x=10\pm 3.536$. Width $=2\times 3.536=$ **7.07 m**. At a height of 7 m the arch is about 7.07 m wide.

d. The widest boat at height 7 m has width 7.07 m, which is the answer to part (c) (the same calculation). So a 7 m high boat can be at most 7.07 m wide.

Answer to Question 20

a. The roots are real and distinct exactly when the discriminant is positive. Discriminant $=k^2-4(k-2)=k^2-4k+8=(k-2)^2+4$. Since $(k-2)^2\geq 0$, the discriminant is at least 4, so it is always positive. Two distinct real roots exist for every k.

b. For $x^2 + bx + c = 0$ the sum of the roots is $-b$ and the product is c. $\alpha + \beta = -k$ and $\alpha\beta = k - 2$.
$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = k^2 - 2(k - 2) = k^2 - 2k + 4 = (k - 1)^2 + 3$. The minimum value is **3**, when $k = 1$.

c. Both roots negative needs sum and product > 0. Sum: $-k$ so $k > 0$. Product: $k - 2 > 0$ so $k > 2$. Both conditions hold when $k > 2$. So both roots are negative when $k > 2$.