Lesson - Graphs and transformations

Year: Year 12

In this chapter you learn the shapes of the standard graphs and how changes to an equation move, stretch or reflect a graph. Examiners reward neat, labelled sketches: mark every intercept, any turning points and any asymptotes.

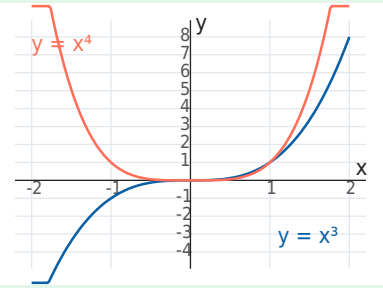
Starter activity

A rollercoaster designer sketches a hill as the curve $y = x^2$ and then wants to move it 3 metres to the right, make it twice as tall and turn it upside down. Sketch the original hill and each stage. At each stage, write down what happens to the point $(1, 1)$. How could you write the final equation without drawing it? Try swapping the order of two of the changes. Do you get the same graph?

Standard graph shapes

Key fact: powers of x

The graph of $y = x^3$ rises from bottom left to top right through the origin. The graph of $y = x^4$ is a $\cup$-shape, flatter near the origin. In general odd powers look like x^3 and even powers look like x^4.

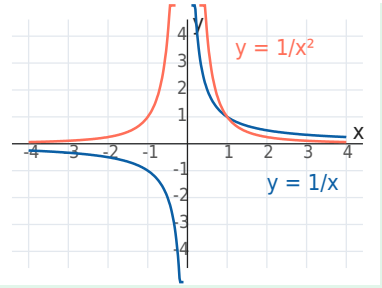


Definition: asymptote

An **asymptote** is a line that a curve gets closer and closer to but never touches. The reciprocal graph $y = \dfrac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

Key fact: reciprocal graphs

$y = \dfrac{1}{x}$ has two branches, in the first and third quadrants. $y = \dfrac{1}{x^2}$ has both branches above the x-axis and is symmetrical about the y-axis.



Sketching polynomials

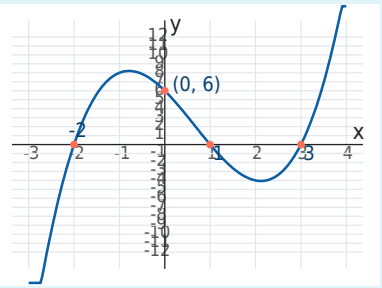
Method: sketching a polynomial

1. Find the roots by factorising and mark them on the x-axis.
2. Find the y-intercept by putting $x = 0$.
3. Decide the end behaviour from the sign of the leading term.
4. A repeated factor $(x - a)^2$ means the graph touches the axis at $x = a$.

Worked example 1

Sketch $y = (x - 1)(x + 2)(x - 3)$.

Roots: $1, -2, 3$. The y-intercept is $(-1)(2)(-3) = 6$. The leading term is x^3, so the graph rises to the right.



Using graphs to solve equations

Rule: intersections give solutions

The solutions of $f(x) = g(x)$ are the x-coordinates where the graphs of $y = f(x)$ and $y = g(x)$ intersect.

Worked example 2

Show that $x^3 = \dfrac{1}{x}$ has exactly two real solutions.

Sketch $y = x^3$ and $y = \dfrac{1}{x}$: they cross at two points, one in each of the first and third quadrants. Algebra confirms $x^4 = 1$, so $x = \pm 1$.

Transformations of $y = f(x)$

Definition: transformation

A **translation** slides a graph, a **stretch** scales it in one direction, and a **reflection** flips it in an axis.

Rule: effect on the graph

Equation	Transformation
$y = f(x) + a$	translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$
$y = f(x + a)$	translation $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
$y = af(x)$	stretch, scale factor a, parallel to the y-axis
$y = f(ax)$	stretch, scale factor $\frac{1}{a}$, parallel to the x-axis
$y = -f(x)$	reflection in the x-axis
$y = f(-x)$	reflection in the y-axis

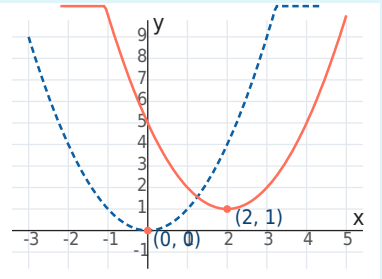
Note: inside or outside?

Changes *inside* the brackets affect x and work "backwards": $f(x - 2)$ moves the graph 2 to the *right*, and $f(2x)$ squashes it towards the y-axis. Changes *outside* the brackets affect y and work as expected.

Worked example 3

The curve $y = x^2$ is translated by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find the new equation.

The new equation is $y = (x - 2)^2 + 1$ and the vertex moves from $(0, 0)$ to $(2, 1)$.



Worked example 4

The point $(4, 6)$ lies on $y = f(x)$. Find its image on $y = f(2x)$, $y = 2f(x)$ and

$y = f(x - 1)$.

On $y = f(2x)$ the x-coordinate is halved: $(2, 6)$. On $y = 2f(x)$ the y-coordinate doubles: $(4, 12)$. On $y = f(x - 1)$ the point moves right 1: $(5, 6)$.

Describing a transformation

Method: describing a transformation

1. If the new equation is a quadratic, complete the square.
2. Compare with the original equation: look inside the brackets for x and outside for y.
3. State the type, the size and the direction (for a stretch give the scale factor and the axis it is parallel to).

Worked example 5

Describe how $y = x^2 - 6x + 11$ is obtained from $y = x^2$.

$x^2 - 6x + 11 = (x - 3)^2 + 2$, so it is $y = x^2$ translated by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Watch out

Do not move $f(x + 3)$ to the right (it moves left). Do not write "stretch by 2" for $f(2x)$: the scale factor is $\frac{1}{2}$. Never let a reciprocal curve touch an axis, and label intercepts with coordinates. Giving only "move" without direction and size earns no marks.

Worked example 6: Sketching a cubic with a repeated root

Sketch $y = x^2(3 - x)$ and find where $y > 0$.

The factor x^2 gives a repeated root at $x = 0$ (the curve touches the axis), and $(3 - x)$ gives a root at $x = 3$ (the curve crosses). The coefficient of x^3 is -1 (negative), so the curve starts high on the left and ends low on the right. Between the roots $y > 0$ for 0 and also for x because $x^2 > 0$ and $3 - x > 0$. So $y > 0$ when x (excluding the touching point $x = 0$ where $y = 0$).

Worked example 7: Sketching a reciprocal curve with a translation

Sketch $y = \dfrac{4}{x + 1} - 2$, showing the asymptotes and the intercepts.

Start from $y = \dfrac{4}{x}$ (two branches in the first and third quadrants). Translate 1 unit left and 2 units down. The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = -2$. When $x = 0$: $y = 4 - 2 = 2$, so the curve meets the y-axis at $(0, 2)$. When

$y = 0$: $\dfrac{4}{x+1} = 2$, $x + 1 = 2$, $x = 1$, so it meets the x-axis at $(1, 0)$. The upper branch passes through both intercepts and the lower branch lies below $y = -2$.

Worked example 8: Combined transformation and order

The point $(2, 5)$ lies on $y = f(x)$. Find its image on the curve $y = 3f(x - 1) + 2$.

Work from the inside outwards. The translation $x \to x - 1$ moves the point 1 to the right: $(3, 5)$. The stretch factor 3 parallel to the y-axis multiplies the y-coordinate: $(3, 15)$. Finally add 2 to the y-coordinate: $(3, 17)$. The order matters: if you added 2 first and then multiplied by 3 you would get 21, which is the image on $y = 3[f(x - 1) + 2]$, a different curve.

Worked example 9: f(ax) compared with af(x)

$f(x) = x^2 - 1$. Find and compare $f(2x)$ and $2f(x)$.

$f(2x) = (2x)^2 - 1 = 4x^2 - 1$. $2f(x) = 2(x^2 - 1) = 2x^2 - 2$. They are different curves. For $f(2x)$ the roots are at $x = \pm\frac{1}{2}$ (half as far from the axis as before) and the y-intercept is still -1. For $2f(x)$ the roots remain at $x = \pm 1$ and the y-intercept is -2. So a stretch parallel to the x-axis changes the x-coordinates, while a stretch parallel to the y-axis changes the y-coordinates.

Worked example 10: Solving an equation using graphs

Show that the equation $x^3 - 3x + 1 = 0$ has three real roots and find an interval for each root.

Let $f(x) = x^3 - 3x + 1$. Then $f(-2) = -1$, $f(-1) = 3$, $f(0) = 1$, $f(1) = -1$, $f(2) = 3$. The sign changes between -2 and -1, between 0 and 1 and between 1 and 2, so there is a root in each interval and, since a cubic has at most three roots, exactly three real roots. More precisely they are approximately -1.88, 0.35 and 1.53. Drawing $y = x^3$ and the line $y = 3x - 1$ on the same axes shows the same three intersections.

Worked example 11: Transforming an exponential graph

Sketch $y = 2^{x-1} + 3$. State the asymptote and the y-intercept, and find where the curve crosses the line $y = 5$.

Start with $y = 2^x$, which passes through $(0, 1)$ and has the x-axis as asymptote. Translate by $(1\backslash 3)$: the asymptote becomes $y = 3$ and the point $(0, 1)$ moves to $(1, 4)$. The y-intercept is $2^{-1} + 3 = 3.5$. The curve is always above the line $y = 3$. To meet $y = 5$ solve $2^{x-1} = 2$, giving $x - 1 = 1$ and $x = 2$. So the curve crosses $y = 5$ at the single point $(2, 5)$.

Worked example 12: Equation of an image under a stretch and a translation

The curve $y = x^3$ is stretched by scale factor 2 parallel to the y-axis and then translated by $(-1\backslash 4)$. Find the equation of the image and its y-intercept.

After the stretch the equation is $y = 2x^3$. Replacing x by $x + 1$ moves the curve one unit left, and adding 4 moves it up: $y = 2(x + 1)^3 + 4$. When $x = 0$: $y = 2 + 4 = 6$. The point of inflection of $y = x^3$ at the origin has moved to $(-1, 4)$, which you can read directly from the equation. Expanding gives $y = 2x^3 + 6x^2 + 6x + 6$.

Note: reading asymptotes from an equation

For a reciprocal graph $y = \dfrac{a}{x - b} + c$ the denominator tells you the vertical asymptote $x = b$ because the function is undefined there, and the added constant tells you the horizontal asymptote $y = c$ because the fraction shrinks towards zero for very large $|x|$. For an exponential graph $y = k \times a^{x-b} + c$ the horizontal asymptote is $y = c$. Whenever you sketch these curves, draw the asymptotes as dashed lines first and then position the branches so that they approach but never touch them. Examiners award marks for the asymptote equations, so write them beside the sketch.

Method: sketching a transformed graph

1. Sketch the original graph and mark key points: intercepts, turning points and asymptotes.
2. Apply the transformations in the order they affect the coordinates: changes inside the bracket (such as $x - 2$ or $2x$) affect the x-coordinates in the opposite sense, changes outside affect the y-coordinates.
3. Plot the new key points and update the asymptotes.
4. Label everything and check that the shape is correct.

Watch out: common graph errors

- Moving $y = f(x + 3)$ to the right. It moves to the left by 3.
- Stretching by factor 2 when the graph is $f(2x)$. The x-coordinates are halved (scale factor $\frac{1}{2}$).
- Forgetting that $y = -f(x)$ changes maxima into minima and vice versa.
- Drawing a curve that crosses its asymptote or a cubic that has a corner.
- Not labelling the intercepts with coordinates, or writing a root as x instead of $(x, 0)$.

Did you know?

The French mathematician René Descartes (1596-1650) is said to have invented the coordinate system while lying in bed watching a fly crawl across the ceiling tiles. By describing the fly\'s position with two numbers he turned geometry into algebra. Every graph you sketch in this chapter descends from that idea, and today computer game

designers use exactly the same transformations (translation, stretches and reflections) to move characters around the screen.

Otto's tip

Ask "is the change next to x or next to y?" Next to x (inside brackets) is backwards and horizontal; outside is normal and vertical.