Worksheet - Graphs and transformations

Year: Year 12 · 20

Show your working.

Question 1

Question 1 – Sketching a cubic from its factors *(Section A · Skills practice)*

Sketch the curve $y = (x + 2)(x - 1)(x - 3)$. Label every point where it crosses the axes.

a. State the roots and the y-intercept. [3 marks]
b. Describe the end behaviour as $x \to \pm\infty$. [2 marks]
c. Solve $(x + 2)(x - 1)(x - 3)$ using your sketch. [2 marks]

Question 2

Question 2 – A repeated root *(Section A · Skills practice)*

Sketch the curve $y = (x - 2)^2(x + 1)$.

a. Explain what the repeated factor $(x - 2)^2$ tells you about the curve at $x = 2$. [2 marks]
b. Find the y-intercept. [1 mark]
c. For which values of x is $y \geq 0$? [2 marks]
d. The curve is translated by $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$. Write down its new equation and find its y-intercept. [3 marks]

Question 3

Question 3 – Reciprocal graphs and asymptotes *(Section A · Skills practice)*

The curve C has equation $y = \dfrac{3}{x - 2} + 1$.

a. Write down the equations of the two asymptotes. [2 marks]
b. Find the coordinates of the points where C meets the axes. [4 marks]
c. Describe the transformation that maps $y = \dfrac{1}{x}$ to C in two steps. [3 marks]

Question 4

Question 4 – Transforming a point *(Section A · Skills practice)*

The point $A(4, 6)$ lies on the curve $y = f(x)$. Write down the coordinates of the image

of A on each of these curves.

Curve	Image of A
$y = f(x) + 2$	?
$y = f(x - 3)$	?
$y = 2f(x)$	?
$y = f(2x)$	?
$y = -f(x)$	?
$y = f(-x)$	?

Question 5

Question 5 – Equation of an image curve *(Section A · Skills practice)*

The curve C has equation $y = x^2 - 3x$. Find the equation of the image of C after each transformation. Give each answer in the form $y = ax^2 + bx + c$.

a. a translation by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ [3 marks]
b. a reflection in the x-axis [1 mark]
c. a stretch with scale factor $\frac{1}{2}$ parallel to the x-axis [2 marks]
d. a reflection in the y-axis [2 marks]

Question 6

Question 6 – Naming the transformation *(Section A · Skills practice)*

Describe fully the single transformation that maps the graph of $y = f(x)$ onto each of these graphs.

a. $y = f(x + 3) - 4$ [2 marks]
b. $y = 3f(x)$ [2 marks]
c. $y = f\left(\frac{x}{2}\right)$ [2 marks]
d. $y = -f(x)$ [1 mark]
e. $y = f(4x)$ [2 marks]

Question 7

Question 7 – The suspension bridge cable *(Section B · Real-life practice)*

The main cable of a footbridge hangs between two towers 60 m apart. The towers are each 24 m tall (measured above the deck). The lowest point of the cable is 4 m above the deck, midway between the towers. Take the left tower at $x = 0$ and let y be the height above the deck.

a. The cable is a transformation of $y = x^2$. Write the equation in the form $y = a(x - p)^2 + q$. [4 marks]
b. Describe the transformations from $y = x^2$ in order. [3 marks]
c. Find the height of the cable 10 m from a tower. Give your answer to 2 decimal places. [2 marks]
d. A support rope is needed where the cable is 10 m above the deck. How far from a tower is it (2 decimal places)? [3 marks]

Question 8

Question 8 – Sharing the journey time *(Section B · Real-life practice)*

A taxi travels 120 km at an average speed of v km/h. The journey time is $t = \dfrac{120}{v}$ hours. When there is a road closure the driver adds a 0.5 hour delay, so $t = \dfrac{120}{v} + 0.5$.

a. Find the journey time at 60 km/h and at 80 km/h without the delay. [2 marks]
b. Describe the transformation from $t = \dfrac{120}{v}$ to $t = \dfrac{120}{v} + 0.5$. State the equations of the asymptotes of each graph. [3 marks]
c. With the delay, find the journey time at 100 km/h. [2 marks]
d. The passenger must arrive within 1.8 hours. Find the minimum average speed with the delay, to 1 decimal place. [3 marks]

Question 9

Question 9 – Pizza night *(Section B · Real-life practice)*

A group of x friends share the cost of a £36 pizza order equally, so each pays $y = \dfrac{36}{x}$ pounds. Delivery costs £3 per person, giving $y = \dfrac{36}{x} + 3$. A £6 discount code reduces the order to £30, giving $y = \dfrac{30}{x} + 3$.

x	3	6	9	12	18
$\dfrac{36}{x}$	12	6	4	3	2
$\dfrac{36}{x} + 3$	15	9	7	6	5

a. Check the table entry for $x = 9$ and add the missing row for $\dfrac{30}{x} + 3$ when $x = 3, 6, 15$. [3 marks]
b. Describe how the graph of $y = \dfrac{36}{x} + 3$ is related to $y = \dfrac{1}{x}$. [2 marks]
c. How many friends are needed for each to pay £5 without the code? [2 marks]

d. How many are needed for each to pay £5 with the code? [2 marks]

Question 10

Question 10 – A hilly cycle route *(Section B · Real-life practice)*

A cycle route follows the ground profile $h = \frac{1}{6}x(x-4)(x-9)$ where x is the horizontal distance in kilometres from the start and h is the height in tens of metres relative to sea level.

a. Show that the route is at sea level at $x = 0$, $x = 4$ and $x = 9$. [2 marks]
b. Sketch the profile for $0 \le x \le 10$. [3 marks]
c. For which values of x in the range is the route above sea level? [2 marks]
d. Find the height at $x = 2$ in metres. [2 marks]

Question 11

Question 11 – Exponential growth sketch *(Section B · Real-life practice)*

A fungus colony covers $y = 3^x$ square centimetres after x days. A second colony covers $y = 3^{x+1}$ square centimetres.

a. Sketch both curves on the same axes for $-2 \le x \le 3$. Include the y-intercepts. [3 marks]
b. Show that $3^{x+1} = 3 \times 3^x$ and describe the transformation from the first to the second in two different ways. [3 marks]
c. State the equation of the asymptote. [1 mark]
d. After how many days is the second colony larger than 100 cm^2? Give your answer to 1 decimal place. [3 marks]

Question 12

Question 12 – How many solutions? *(Section B · Real-life practice)*

The curve $y = x^2(x-3)$ models the water level in a tank over time. The table gives some values.

x	-1	0	1	2	3	4
y	-4	0	-2	-4	0	16

a. Check the values of y at $x = -1$ and $x = 2$. [2 marks]
b. Sketch the curve, marking the turning points. [3 marks]
c. The line $y = k$ is drawn. State the number of solutions of $x^2(x-3) = k$ when $k = -3$, $k = -4$, $k = 0$ and $k = 5$. [4 marks]
d. Show that $x^3 - 3x^2 + 4 = 0$ factorises as $(x+1)(x-2)^2$ and say what this means for $k = -4$. [3 marks]

Question 13

Question 13 – Reading a cooling curve *(Section B · Real-life practice)*

The temperature of a drink after t minutes is modelled by $T = 20 + 60 \times 2^{-t/5}$ degrees Celsius.

t (min)	0	5	10	15
T (°C)	80	50	35	27.5

a. Verify the table at $t = 5$ and $t = 15$. [2 marks]
b. State the horizontal asymptote and explain its meaning in the context. [2 marks]
c. Describe the transformations that map $y = 2^t$ to T in order. [4 marks]
d. The drink is safe to hold when it is below 40°C. After how many minutes (to 1 decimal place)? [3 marks]

Question 14

Question 14 – Transforming key points *(Section C · Problem solving)*

The diagram of $y = f(x)$ (not shown to scale) has these key points: it crosses the x-axis at $A(-3, 0)$ and $D(3, 0)$, has a maximum at $B(-1, 4)$ and a minimum at $C(1, -2)$.

Point	Coordinates
A	$(-3, 0)$
B (maximum)	$(-1, 4)$
C (minimum)	$(1, -2)$
D	$(3, 0)$

a. For $y = f(x) + 3$ write down the coordinates of B and C. [2 marks]
b. For $y = f(x - 2)$ write down the coordinates of all four points. [4 marks]
c. For $y = 2f(x)$ write down the new turning points. [2 marks]
d. For $y = f(2x)$ write down the images of A and D. [2 marks]
e. For $y = -f(x)$ write down the new turning points and say which is a maximum. [3 marks]

Question 15

Question 15 – A cubic and its translations *(Section C · Problem solving)*

$f(x) = x^3 - 4x$.

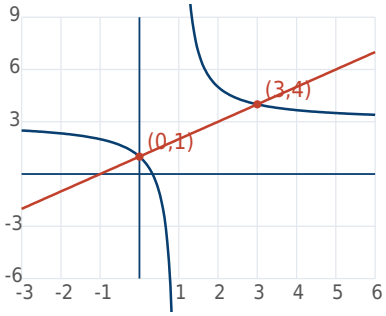
a. Show that $f(x) = x(x - 2)(x + 2)$ and sketch $y = f(x)$. [3 marks]
b. On the same axes sketch $y = f(x - 1)$, marking the x-intercepts. [3 marks]
c. Find the y-intercept of $y = f(x - 1)$. [2 marks]
d. The curve $y = f(x + 2)$ passes through the origin. Write down its equation in factorised form and verify that it passes through $(0, 0)$. [3 marks]

Question 16

Question 16 – A hyperbola and a line *(Section C · Problem solving)*

The curve C has equation $y = \dfrac{2}{x-1} + 3$ and the line L has equation $y = x + 1$.

a. Write down the equations of the asymptotes of C. [2 marks]
b. Find where C crosses the axes. [3 marks]
c. Show that the x-coordinates of the points where L meets C satisfy $x^2 - 3x = 0$. [4 marks]
d. Find the coordinates of the points of intersection. [2 marks]



Question 17

Question 17 – Equations from sketches *(Section C · Problem solving)*

Find the equation of each curve described below.

a. A quadratic curve with minimum point $(3, -4)$ that passes through $(0, 5)$. [4 marks]
b. A cubic curve that touches the x-axis at $x = -2$, crosses it at $x = 1$ and passes through $(0, -8)$. [4 marks]
c. Find the height of the cubic in part (b) when $x = 2$. [1 mark]
d. The graph of part (a) is reflected in the x-axis. Write down the new equation and the coordinates of the maximum point. [2 marks]

Question 18

Question 18 – Designing a window *(Section C · Problem solving)*

An architect designs an arched window. The top edge of the glass is the curve $y = 9 - x^2$ for $-3 \le x \le 3$ (in feet), with the straight bottom edge on the x-axis.

a. Show that the curve is the image of $y = x^2$ after a reflection in the x-axis followed by a translation. State the vector. [3 marks]
b. Find the height of the window at $x = 1$ and at $x = 2$. [2 marks]
c. The architect considers the wider arch $y = 9 - \dfrac{x^2}{4}$. Describe the transformation from $y = 9 - x^2$ to this curve and find the new width of the window at the bottom.

[4 marks]

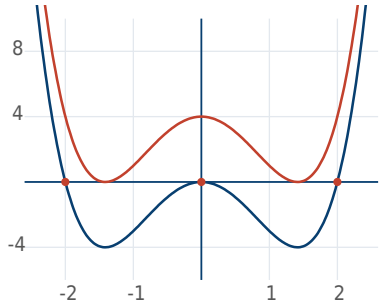
d. Both windows are 9 ft high in the middle. Find where the two curves meet and at what height. [4 marks]

Question 19

Question 19 – A quartic and its symmetry *(Section C · Problem solving)*

$f(x) = x^4 - 4x^2$.

a. Factorise $f(x)$ completely and sketch the curve. [4 marks]
b. Show that $f(-x) = f(x)$ and say what this means for the graph. [2 marks]
c. Solve $f(x)$. [3 marks]
d. The curve is translated by $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$. Show that the new equation can be written $y = (x^2 - 2)^2$ and state the coordinates of the points where it touches the x-axis. [4 marks]



Question 20

Question 20 – Challenge: reversing a transformation *(Section D · Challenge)*

The curve $y = f(x)$ is translated by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ to give the curve $y = x^2 - 6x + 12$.

a. Find $f(x)$. [4 marks]
b. Show that neither curve crosses the x-axis. [3 marks]
c. The curve $y = f(x) - k$ is a translation of $y = f(x)$ vertically. Find the set of values of k for which it meets the x-axis at two distinct points. [4 marks]
d. Find the translation vector that moves $y = f(x)$ so that its turning point lies on the x-axis. [2 marks]