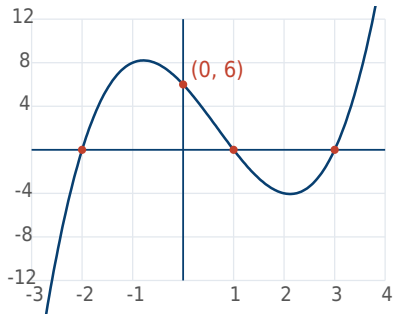
Answers - Graphs and transformations

Year: Year 12

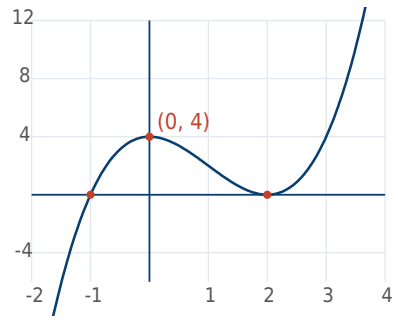
Answer to Question 1

a. Each factor gives a root when it equals zero. The curve crosses the x-axis at $x = -2$, $x = 1$ and $x = 3$. When $x = 0$: $y = 2 \times (-1) \times (-3) = 6$, so the y-intercept is $(0, 6)$.
b. The coefficient of x^3 is positive, so $y \to +\infty$ as $x \to +\infty$ and $y \to -\infty$ as $x \to -\infty$. So the curve comes up from the bottom left and goes off to the top right.
c. Read from the sketch where the curve lies below the x-axis. The curve is below the x-axis for x and for 1. So the solution is x **or** 1.



Answer to Question 2

a. A squared factor gives a repeated root. The curve **touches** the x-axis at $(2, 0)$ but does not cross it: there is a turning point on the axis. The curve changes direction there instead of crossing.
b. Substitute $x = 0$. $y = (-2)^2 \times 1 = \mathbf{4}$. So the y-intercept is $(0, 4)$.
c. $(x - 2)^2 \geq 0$ always, so $y \geq 0$ when $x + 1 \geq 0$ or when $x = 2$: $x \geq -1$ (and $x = 2$ is included in this).
d. A translation by $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ subtracts 4 from every y-value. New equation $y = (x - 2)^2(x + 1) - 4$. The y-intercept is $4 - 4 = \mathbf{0}$, i.e. the curve passes through the origin.



Answer to Question 3

a. The denominator is zero when $x = 2$, so one asymptote is $x = 2$. As $x \to \pm\infty$, $\dfrac{3}{x-2} \to 0$ so $y \to 1$: the other asymptote is $y = 1$. The curve approaches these two lines but never meets them.

b. Find the intercepts by setting $x = 0$ and then $y = 0$. Put $x = 0$: $y = \dfrac{3}{-2} + 1 = -\dfrac{1}{2}$. Point $(0, -\frac{1}{2})$. Put $y = 0$: $\dfrac{3}{x-2} = -1$, $x - 2 = -3$, $x = -1$. Point $(-1, 0)$. So C meets the axes at $(0, -\frac{1}{2})$ and $(-1, 0)$.

c. Compare $y = \dfrac{3}{x-2} + 1$ with $y = \dfrac{1}{x}$: the 3 multiplies the output, the -2 moves the graph right and the $+1$ moves it up. First a stretch with scale factor 3 parallel to the y-axis ($y = \frac{3}{x}$), then a translation by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The asymptotes move from $x = 0$, $y = 0$ to $x = 2$, $y = 1$.

Answer to Question 4

Changes outside the function act on the y-coordinate in the expected way; changes inside the function act on the x-coordinate in the opposite way.

Curve	Image of A	Reason
$y = f(x) + 2$	$(4, 8)$	y-values increase by 2
$y = f(x - 3)$	$(7, 6)$	shift 3 right: $x + 3$
$y = 2f(x)$	$(4, 12)$	y-values doubled
$y = f(2x)$	$(2, 6)$	x-values halved
$y = -f(x)$	$(4, -6)$	reflection in the x-axis
$y = f(-x)$	$(-4, 6)$	reflection in the y-axis

For example, $f(2x)$ halves the x-coordinate because $2 \times 2 = 4$ gives the original input $x = 4$, and $f(x - 3)$ moves the point 3 to the right because $7 - 3 = 4$.

Answer to Question 5

a. A translation by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ replaces x by $x - 2$ and subtracts 1 from the whole function. Replace x by $(x - 2)$ and subtract 1:
$y = (x-2)^2 - 3(x-2) - 1 = x^2 - 4x + 4 - 3x + 6 - 1 = x^2 - 7x + 9$.

b. A reflection in the x-axis changes the sign of every y-value. Multiply y by -1:
$y = -x^2 + 3x$.

c. A stretch of scale factor $\frac{1}{2}$ parallel to the x-axis replaces x by $2x$:

$y = (2x)^2 - 3(2x) = 4x^2 - 6x$. Each x-coordinate of the image is half the original one.

d. A reflection in the y-axis replaces x by $-x$. Replace x by $-x$: $y = (-x)^2 - 3(-x) = x^2 + 3x$.

Answer to Question 6

a. Inside the bracket, $+3$ moves the graph left; outside, -4 moves it down. A translation by the vector $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$ (3 left, 4 down).

b. Multiplying the whole function by 3 multiplies every y-coordinate by 3. A stretch with scale factor 3 parallel to the y-axis.

c. Replacing x by $\frac{x}{2}$ acts on the input. A stretch with scale factor 2 parallel to the x-axis (the x-coordinates double).

d. Every y-coordinate changes sign. A reflection in the x-axis.

e. Replacing x by $4x$ divides each x-coordinate by 4. A stretch with scale factor $\frac{1}{4}$ parallel to the x-axis.

Answer to Question 7

a. The lowest point is the vertex, midway between the towers at $x = 30$; use the top of the left tower, $(0, 24)$, to find a. Lowest point $(30, 4)$, so $y = a(x - 30)^2 + 4$. At $x = 0$, $y = 24$: $900a + 4 = 24$, $a = \dfrac{20}{900} = \dfrac{1}{45}$. So $y = \dfrac{1}{45}(x - 30)^2 + 4$.

b. Start from $y = x^2$, apply the vertical stretch, then move the vertex to $(30, 4)$. A stretch with scale factor $\frac{1}{45}$ parallel to the y-axis, then a translation by $\begin{pmatrix} 30 \\ 4 \end{pmatrix}$.

c. 10 m from the left tower means $x = 10$ (by symmetry, $x = 50$ gives the same height). $x = 10$: $y = \dfrac{(-20)^2}{45} + 4 = \dfrac{400}{45} + 4 = 8.89 + 4 =$ **12.89 m**.

d. Set $y = 10$ and solve for x. $\dfrac{(x - 30)^2}{45} = 6$, so $(x - 30)^2 = 270$, $x - 30 = \pm 16.43$, giving $x = 13.57$ (or 46.43). About **13.57 m** from the left tower. By symmetry, the rope is also 13.57 m from the right tower.

Answer to Question 8

a. Substitute each speed into $t = \dfrac{120}{v}$. 60 km/h: $120 \div 60 =$ **2 hours**. 80 km/h: $120 \div 80 =$ **1.5 hours**.

b. Adding 0.5 to the output moves the graph up 0.5. A translation by $\begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$ (up by

0.5). Original asymptotes: $v = 0$ and $t = 0$. New asymptotes: $v = 0$ and $t = 0.5$.

c. Substitute $v = 100$ into the formula with the delay. $\frac{120}{100} + 0.5 =$ **1.7 hours**.

d. Form an inequality; since $v > 0$ we can multiply through by v without changing the direction. $\frac{120}{v} + 0.5 \leq 1.8$ so $\frac{120}{v} \leq 1.3$, $v \geq \frac{120}{1.3} =$ **92.3 km/h**. The minimum average speed is 92.3 km/h.

Answer to Question 9

a. Substitute each value of x into the formula. $x = 9$: $36 \div 9 = 4$, $4 + 3 = 7$ ✓. With the code: $x = 3$: $10 + 3 = 13$; $x = 6$: $5 + 3 = 8$; $x = 15$: $2 + 3 = 5$.

b. Compare $\frac{36}{x} + 3$ with $\frac{1}{x}$. It is a stretch with scale factor 36 parallel to the y-axis followed by a translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

c. Set the cost per person equal to 5 and solve. $\frac{36}{x} + 3 = 5$ gives $\frac{36}{x} = 2$, $x =$ **18**. 18 friends are needed without the code.

d. Repeat with the cheaper order. $\frac{30}{x} + 3 = 5$ gives $\frac{30}{x} = 2$, $x =$ **15**. With the code, 15 friends are enough.

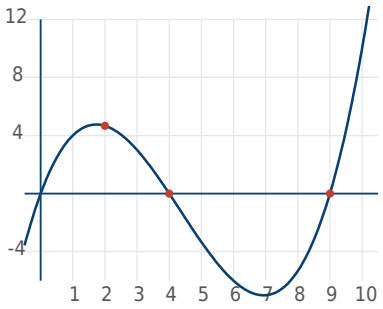
Answer to Question 10

a. Substitute each value of x into h. Each factor is zero in turn: $x = 0$, $x = 4$, $x = 9$.

b. It is a cubic with positive leading coefficient: rises from the origin to a maximum between 0 and 4, falls below the axis between 4 and 9 and rises again after 9. At $x = 10$: $h = \frac{1}{6} \times 10 \times 6 \times 1 = 10$.

c. Use the sketch: the curve is above the axis between the first two roots and after the third. $h > 0$ for 0 and $x > 9$ (up to $x = 10$).

d. Substitute $x = 2$ and convert from tens of metres to metres. $h(2) = \frac{1}{6} \times 2 \times (-2) \times (-7) = \frac{28}{6} = 4.67$ tens of metres, i.e. about **46.7 m**.



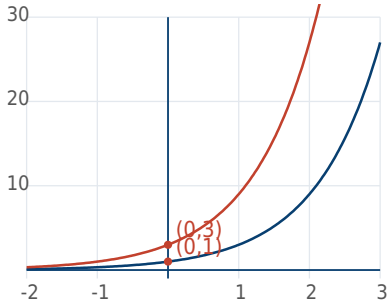
Answer to Question 11

a. Find a few points on each curve, such as the y-intercepts. Both curves are increasing and positive. $y = 3^x$ passes through $(0, 1)$ and $(1, 3)$. $y = 3^{x+1}$ passes through $(0, 3)$ and $(-1, 1)$. For $x \leq -2$ both are very close to the axis.

b. Use the index law $a^{m+n} = a^m \times a^n$. $3^{x+1} = 3^x \times 3^1 = 3 \times 3^x$. So the second graph is a **translation by** $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ or equivalently a **stretch with scale factor 3 parallel to the y-axis**.

c. The asymptote is $y = 0$ for both.

d. Take logarithms of both sides. $3^{x+1} = 100$: $x + 1 = \dfrac{\log 100}{\log 3} = 4.19$, so $x = 3.19$. After about **3.2 days**.



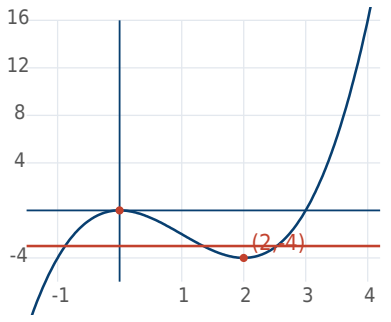
Answer to Question 12

a. Substitute the values into $y = x^2(x - 3)$. $x = -1$: $1 \times (-4) = -4$ ✓. $x = 2$: $4 \times (-1) = -4$ ✓.

b. Roots at $x = 0$ (touching, repeated) and $x = 3$. Local maximum at $(0, 0)$ and local minimum at $(2, -4)$. The curve falls to $-\infty$ on the left and rises on the right.

c. $k = -3$: **3 solutions**. $k = -4$: **2 distinct solutions** ($x = -1$ and the repeated $x = 2$). $k = 0$: **2 solutions** ($x = 0$ repeated and $x = 3$). $k = 5$: **1 solution** (the curve only reaches 5 once, for $x > 3$).

d. $(x + 1)(x - 2)^2 = (x + 1)(x^2 - 4x + 4) = x^3 - 3x^2 + 4$ ✓. So $x^2(x - 3) = -4$ has the solution $x = -1$ and a repeated solution $x = 2$ where the line touches the minimum.



Answer to Question 13

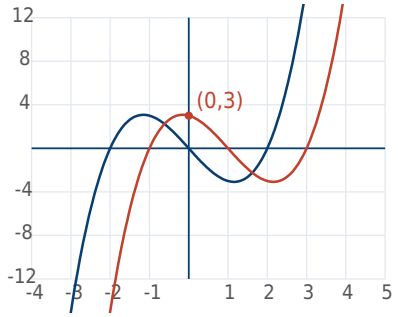
a. Substitute the values into the formula. $t = 5$: $20 + 60 \times 2^{-1} = 20 + 30 = 50$ ✓. $t = 15$: $20 + 60 \times 2^{-3} = 20 + 7.5 = 27.5$ ✓.
b. As $t \to \infty$, $2^{-t/5} \to 0$. $T = 20$. The temperature approaches the room temperature (20°C) but never goes below it.
c. Reflection in the y-axis (2^{-t}), then a stretch with scale factor 5 parallel to the t-axis ($2^{-t/5}$), then a stretch with scale factor 60 parallel to the T-axis, then a translation by $\begin{pmatrix} 0 \\ 20 \end{pmatrix}$.
d. Solve the inequality with logarithms. $20 + 60 \times 2^{-t/5}$ gives $2^{-t/5}$, so $\frac{t}{5} > \log_2 3 = 1.585$, $t > 7.9$. After about **7.9 minutes**. The drink is safe to hold after about 7.9 minutes.

Answer to Question 14

a. Adding 3 to the function moves every point up 3. $B \to (-1, 7)$, $C \to (1, 1)$.
b. Replacing x by $x - 2$ moves the graph 2 units right. Add 2 to each x-coordinate: $A \to (-1, 0)$, $B \to (1, 4)$, $C \to (3, -2)$, $D \to (5, 0)$.
c. Multiplying the function by 2 doubles every y-coordinate. Double the y-coordinates: $B \to (-1, 8)$, $C \to (1, -4)$.
d. Replacing x by $2x$ halves every x-coordinate. Halve the x-coordinates: $A \to (-\frac{3}{2}, 0)$ and $D \to (\frac{3}{2}, 0)$.
e. A reflection in the x-axis turns a maximum into a minimum and vice versa. Change the sign of y: $B \to (-1, -4)$ and $C \to (1, 2)$. The new maximum is $(1, 2)$ (the image of C) and the new minimum is $(-1, -4)$.

Answer to Question 15

a. Take out the common factor x, then use the difference of two squares. $x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$. The curve crosses the axis at -2, 0 and 2, rising on the right and falling on the left.
b. Replacing x by $x - 1$ moves the graph 1 unit right. Translating 1 to the right moves the roots to -1, 1 and 3: $y = (x - 1)(x - 3)(x + 1)$.
c. The y-intercept of $y = f(x - 1)$ is the value at $x = 0$, which is $f(-1)$. $f(-1) = -1 + 4 = \mathbf{3}$, so $(0, 3)$.
d. Replace x by $x + 2$ in the factorised form. $y = f(x + 2) = (x + 2)x(x + 4)$. At $x = 0$: $y = 2 \times 0 \times 4 = 0$ ✓. Its roots are -4, -2, 0.



Answer to Question 16

a. The denominator is zero at $x = 1$, and as $x \to \pm\infty$ the fraction tends to 0 so $y \to 3$. $x = 1$ and $y = 3$. These are the vertical and horizontal asymptotes of C.

b. Set $x = 0$ for the y-intercept and $y = 0$ for the x-intercept. y-intercept: $\frac{2}{-1} + 3 = 1$, point $(0, 1)$. x-intercept: $\frac{2}{x-1} = -3$, $x = 1 - \frac{2}{3} = \frac{1}{3}$, point $(\frac{1}{3}, 0)$. So C crosses the axes at $(0, 1)$ and $(\frac{1}{3}, 0)$.

c. Set the two expressions for y equal and multiply through by $x - 1$. $\frac{2}{x-1} + 3 = x + 1 \Rightarrow \frac{2}{x-1} = x - 2 \Rightarrow 2 = (x-2)(x-1) = x^2 - 3x + 2 \Rightarrow x^2 - 3x = 0$. This is the required equation.

d. Factorise and substitute each root into the line $y = x + 1$. $x(x - 3) = 0$ gives $x = 0$ and $x = 3$. Points $(0, 1)$ and $(3, 4)$. Check on C: $\frac{2}{3-1} + 3 = 4$ ✓.

Answer to Question 17

a. A quadratic with minimum (p, q) can be written $y = a(x - p)^2 + q$; use the given point to find a. Vertex form: $y = a(x - 3)^2 - 4$. At $(0, 5)$: $9a - 4 = 5$, $a = 1$. So $y = (x - 3)^2 - 4 = x^2 - 6x + 5$. Check: $0 - 0 + 5 = 5$ ✓.

b. Touching the axis at $x = -2$ gives a squared factor $(x + 2)^2$; crossing at $x = 1$ gives a factor $(x - 1)$. $y = a(x + 2)^2(x - 1)$. At $x = 0$: $a \times 4 \times (-1) = -8$, $a = 2$. So $y = 2(x + 2)^2(x - 1)$. Check: $2 \times 4 \times (-1) = -8$ ✓.

c. Substitute $x = 2$ into $y = 2(x + 2)^2(x - 1)$. $y = 2 \times 16 \times 1 = \mathbf{32}$. The height of the cubic at $x = 2$ is 32.

d. A reflection in the x-axis changes the sign of y, so the minimum becomes a maximum. New equation $y = -(x - 3)^2 + 4$ (or $y = -x^2 + 6x - 5$), maximum at $(3, 4)$. The vertex $(3, -4)$ maps to $(3, 4)$.

Answer to Question 18

a. A reflection in the x-axis changes x^2 into $-x^2$; a translation upwards then adds 9.

Reflect $y = x^2$ in the x-axis: $y = -x^2$. Translate by $\begin{pmatrix} 0 \\ 9 \end{pmatrix}$: $y = 9 - x^2$.

b. Substitute into $y = 9 - x^2$. $x = 1$: $9 - 1 =$ **8 ft**. $x = 2$: $9 - 4 =$ **5 ft**.

c. Replace x by $\frac{x}{2}$: a stretch with scale factor 2 parallel to the x-axis. The new bottom edge: $9 - \frac{x^2}{4} = 0$, $x^2 = 36$, $x = \pm 6$, so the width is **12 ft** (the old width was 6 ft).

d. Set the two expressions equal. $9 - x^2 = 9 - \frac{x^2}{4} \Rightarrow \frac{3}{4}x^2 = 0 \Rightarrow x = 0$. They meet only at the top, $(0, 9)$.

Answer to Question 19

a. Take out x^2 and use the difference of two squares. $x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x - 2)(x + 2)$. The curve crosses the axis at $x = \pm 2$ and touches it at the origin. The x^4 coefficient is positive, so both ends go up. There is a maximum at $(0, 0)$ and minima at $(\pm\sqrt{2}, -4)$.

b. $f(-x) = (-x)^4 - 4(-x)^2 = x^4 - 4x^2 = f(x)$. The graph is symmetrical about the y-axis.

c. $x^2 > 0$ for $x \neq 0$, so $f(x)$ when $(x - 2)(x + 2)$, i.e. -2**,** $x \neq 0$.

d. Adding 4 to $f(x)$ moves the curve up 4 units. New equation $y = x^4 - 4x^2 + 4 = (x^2 - 2)^2$. It touches the x-axis where $x^2 = 2$, i.e. at $(\sqrt{2}, 0)$ **and** $(-\sqrt{2}, 0)$.

Answer to Question 20

a. The given curve is $g(x) = x^2 - 6x + 12$; undo the translation. The image is $g(x) = f(x - 2) - 1$. So $f(x - 2) = g(x) + 1 = x^2 - 6x + 13$. Replace x by $x + 2$: $f(x) = (x + 2)^2 - 6(x + 2) + 13 = x^2 + 4x + 4 - 6x - 12 + 13 = x^2 - 2x + 5$. Check: $f(x - 2) - 1 = (x - 2)^2 - 2(x - 2) + 5 - 1 = x^2 - 6x + 12$ ✓.

b. A quadratic crosses the x-axis only if its discriminant $b^2 - 4ac$ is at least 0. f: discriminant $4 - 20 = -16$. g: discriminant $36 - 48 = -12$. Neither has real roots, so neither crosses the axis.

c. Two distinct points need a positive discriminant. $f(x) - k = x^2 - 2x + 5 - k$ has discriminant $4 - 4(5 - k) = 4k - 16 > 0$ when $k > 4$. So the curve meets the axis twice when $k > 4$.

d. Complete the square to find the turning point. $f(x) = (x - 1)^2 + 4$ has its minimum at $(1, 4)$. Translate by $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ to place it on the x-axis.