Test answers - Exponentials and logarithms

Year: Year 12 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

The laws used are $k\log_a x = \log_a x^k$, $\log_a x + \log_a y = \log_a xy$ and $\log_a x - \log_a y = \log_a \dfrac{x}{y}$.

Use the power law $2\log_5 3 = \log_5 9$, then the multiplication and division laws.

$\log_5 \dfrac{9 \times 50}{18}$ (2) $= \log_5 25$ (1) $= 2$ (1).

Since $5^2 = 25$, the expression equals 2.

Check with a calculator: $2\log_5 3 + \log_5 50 - \log_5 18 \approx 1.365 + 2.431 - 1.796 = 2.000$ ✓.

Do not write $2\log_5 3$ as $\log_5 6$: the power law squares the 3, it does not double it. The final mark needs the exact value 2, not a decimal approximation.

Answer to Question 2 (4 marks)

Notice that $4^x = (2^x)^2$, so substituting $y = 2^x$ gives a quadratic $y^2 = 7y - 12$.

Let $y = 2^x$: $y^2 - 7y + 12 = 0$ (1), $(y-3)(y-4) = 0$ (1). $2^x = 4$ gives $x = 2$ (1); $2^x = 3$ gives $x = \log_2 3 = 1.58$ (1).

So $x = 2$ or $x = \log_2 3 \approx 1.58$. Both are valid because 2^x can be 3 or 4.

Check $x = 2$: $4^2 = 16$ and $7 \times 4 - 12 = 16$ ✓.

Check $x = \log_2 3$: $4^x = (2^x)^2 = 9$ and $7 \times 3 - 12 = 9$ ✓. The exact form $x = \dfrac{\ln 3}{\ln 2}$ is equally acceptable for the final mark.

Answer to Question 3 (4 marks)

Both $2x + 1$ and $x - 1$ must be positive for the logarithms to exist, so any solution must satisfy $x > 1$.

Combine with the multiplication law: $\log_3[(2x+1)(x-1)] = 2$, so $(2x+1)(x-1) = 3^2$.

$(2x+1)(x-1) = 9$ (1), $2x^2 - x - 10 = 0$ (1), $(2x-5)(x+2) = 0$ (1). $x = 2.5$ since $x = -2$ is invalid (1).

$x = -2$ is rejected because $\log_3(x-1)$ would be the logarithm of a negative number. So $x = 2.5$.

Check: $\log_3(2 \times 2.5 + 1) + \log_3(2.5 - 1) = \log_3 6 + \log_3 1.5 = \log_3 9 = 2$ ✓.

The last mark is withheld if $x = -2$ is not explicitly rejected.

Answer to Question 4 (4 marks)

Take logarithms of $y = ab^x$ to get a straight line with intercept $\log a$ and gradient $\log b$.

$\log y = \log a + x \log b$ (1). $\log a = 1.3$, $a = 19.9$ (1). $\log b = 0.2$, $b = 1.58$ (2).

So $a = 10^{1.3} = 19.9$ and $b = 10^{0.2} = 1.58$, giving the model $y = 19.9 \times 1.58^x$.

Check: when $x = 0$, $y = a = 19.9$ and $\log_{10} 19.9 = 1.30$, which is the intercept ✓.

A common slip is to take $a = 1.3$ instead of $10^{1.3}$: the intercept is $\log_{10} a$, not a itself. Using $\ln$ in place of $\log_{10}$ here would give wrong values of both constants.

Answer to Question 5 (4 marks)

a. Substitute $t = 4$. $80e^{-0.6} = 43.9$ mg (1) After 4 hours the mass is 43.9 mg. Check: $e^{-0.6} \approx 0.549$ and $80 \times 0.549 \approx 43.9$ ✓.

b. The half-life is the time for the mass to fall from 80 to 40 mg. $e^{-0.15t} = \frac{1}{2}$, $t = \dfrac{\ln 2}{0.15} = 4.62$ h (2) The half-life is 4.62 hours. This half-life does not depend on the initial dose of 80 mg.

c. $\frac{5}{80} = \frac{1}{16}$; take logarithms and reverse the inequality. $e^{-0.15t} < \frac{1}{16}$, $t > \dfrac{\ln 16}{0.15} = 18.5$ h (1) The mass falls below 5 mg after about 18.5 hours. Note that 5 mg is $\frac{1}{16}$ of 80 mg, so this is exactly four half-lives: $4 \times 4.62 = 18.5$ h ✓.