Lesson - Exponentials and logarithms

Year: Year 12

Exponential functions model growth and decay, and logarithms are their inverses. Together they let you solve equations where the unknown is in a power, and turn curved data into straight lines.

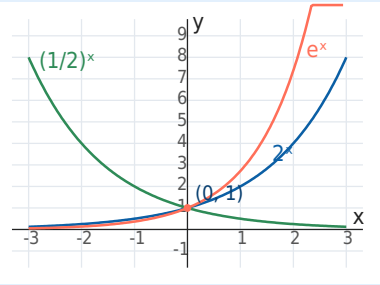
Starter activity

Fold a sheet of paper in half, then in half again, and keep going. The number of layers doubles each time: 1, 2, 4, 8, \dots After how many folds would the stack be taller than you, if a sheet is 0.1 mm thick? Try to guess first, then calculate with powers of 2. Now turn the question around: how many doublings are needed to reach one million? Which operation answers questions of the form "to what power must I raise 2 to get this number"?

Exponential functions

Definition: exponential function

An **exponential function** is $y = a^x$ with $a > 0$, $a \neq 1$. Its graph always passes through $(0, 1)$ and has the x-axis as an asymptote. If $a > 1$ it is increasing (growth); if 0 it is decreasing (decay).



Key fact: the number e

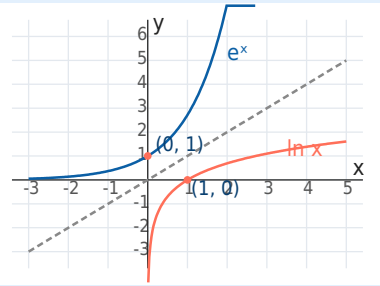
$e \approx 2.718$ is the number for which the gradient of $y = e^x$ equals its height: $\frac{d}{dx}(e^x) = e^x$. More generally $\frac{d}{dx}(e^{kx}) = ke^{kx}$. The gradient of $y = e^{3x}$ at $x = 0$ is therefore 3.

Logarithms

Definition: logarithm

$\log_a b = c$ means $a^c = b$. So $\log_2 32 = 5$ because $2^5 = 32$. The **natural logarithm** $\ln x$ means $\log_e x$, the inverse of e^x: $\ln(e^x) = x$ and $e^{\ln x} = x$. Their graphs are reflections in

the line $y = x$.



Laws of logarithms

Rule: laws of logarithms

- $\log_a x + \log_a y = \log_a(xy)$
- $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$
- $k\log_a x = \log_a(x^k)$
- $\log_a 1 = 0$, $\log_a a = 1$, $\log_a\left(\frac{1}{x}\right) = -\log_a x$
- Change of base: $\log_a b = \frac{\log_c b}{\log_c a}$

Worked example 1

Write $2\log 3 + \log 4$ as a single logarithm.

$\log 9 + \log 4 = \log 36$.

Watch out

$\log(x+y) \neq \log x + \log y$: the law is for products, not sums. Also $\frac{\log x}{\log y} \neq \log x - \log y$. Do not mix up the base and the argument in $\log_a b$.

Solving equations

Method: solving $a^x = b$

1. Take logs of both sides: $x\log a = \log b$.
2. Divide: $x = \frac{\log b}{\log a}$.
3. For the exact value leave it as $\frac{\ln b}{\ln a}$; otherwise round only at the end.

Worked example 2

Solve $3^x = 20$ and $3^x = 2^{x+1}$.

$3^x = 20 \Rightarrow x = \dfrac{\ln 20}{\ln 3} = 2.73$. For the second: $x \ln 3 = (x+1)\ln 2$, $x(\ln 3 - \ln 2) = \ln 2$, so $x = \dfrac{\ln 2}{\ln 1.5} = 1.71$.

Method: equations with logs

1. Combine the logs into one log.
2. Remove the log by writing the equation as a power.
3. Solve, then *check* that every solution makes each log argument positive.

Worked example 3

Solve $\log_2(x+3) + \log_2(x-3) = 4$.

$\log_2(x^2 - 9) = 4 \Rightarrow x^2 - 9 = 16 \Rightarrow x = \pm 5$. Rejecting $x = -5$ (negative arguments) leaves $x = 5$.

Worked example 4

Solve $2^{2x} - 6 \times 2^x + 8 = 0$.

Let $y = 2^x$. Then $y^2 - 6y + 8 = 0$, so $y = 2$ or 4. Hence $x = 1$ or $x = 2$.

Straight-line graphs from curved data

Rule: reducing to a straight line

For $y = ax^n$: $\log y = n \log x + \log a$, so $\log y$ against $\log x$ is a straight line with gradient n and intercept $\log a$. For $y = ab^x$: $\log y = x \log b + \log a$, so $\log y$ against x is straight with gradient $\log b$.

Worked example 5

A graph of $\log_{10} y$ against x passes through $(0, 1)$ and $(2, 1.954)$. Find y in terms of x.

Intercept $1 = \log a \Rightarrow a = 10$. Gradient $\dfrac{0.954}{2} = 0.477 = \log b \Rightarrow b = 3$. So $y = 10 \times 3^x$.

Worked example 6

$y = ax^n$ gives $y = 12$ at $x = 2$ and $y = 192$ at $x = 8$. Find a and n.

Divide: $16 = 4^n$, so $n = 2$, then $a = 3$. A log-log graph would show a line of gradient 2.

Growth and decay models

Definition: exponential growth and decay

Continuous growth is modelled by $P = P_0 e^{kt}$ with $k > 0$, and decay by $P = P_0 e^{-kt}$; P_0 is the initial value. The **doubling time** solves $e^{kt} = 2$, so $t = \dfrac{\ln 2}{k}$; the **half-life** is also $\dfrac{\ln 2}{k}$ for decay.

Worked example 7

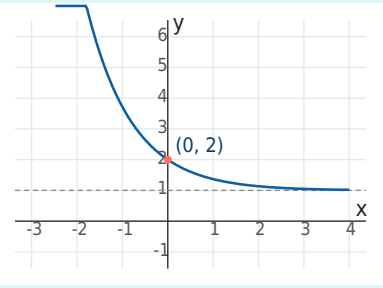
$P = 200e^{0.03t}$. Find $P(10)$ and the doubling time.

$P(10) = 200e^{0.3} \approx 270$, and the population doubles after $\dfrac{\ln 2}{0.03} = 23.1$ years.

Worked example 8

£2000 earns 3% per year. When does it first exceed £3000?

Solve $1.03^n > 1.5$: $n > \dfrac{\ln 1.5}{\ln 1.03} = 13.7$, so 14 full years. Dividing by $\ln 1.03$ is safe because it is positive; dividing by a negative log would reverse the inequality.



Worked example 9

The mass of a sample is $N = 800e^{-0.04t}$ (grams, t in days). Find the half-life and the mass after 20 days.

$e^{-0.04t} = \frac{1}{2}$, so $t = \dfrac{\ln 2}{0.04} = 17.3$ days. After 20 days $800e^{-0.8} \approx 359$ g.

Watch out

Reject solutions that make a log argument negative or zero. Do not round too early in $\dfrac{\ln b}{\ln a}$. For modelling, define what $t = 0$ means and give units.

Worked example 10: Using the laws to simplify

Given that $\log_2 a = p$ and $\log_2 b = q$, write $\log_2 \dfrac{8a^2}{\sqrt{b}}$ in terms of p and q.

$\log_2 \dfrac{8a^2}{\sqrt{b}} = \log_2 8 + \log_2 a^2 - \log_2 b^{1/2} = 3 + 2\log_2 a - \frac{1}{2}\log_2 b = 3 + 2p - \frac{1}{2}q$. For a numerical check take $a = 4$ and $b = 16$, so $p = 2$ and $q = 4$: the expression $\dfrac{8 \times 16}{4} = 32$ has $\log_2 32 = 5$, and $3 + 2 \times 2 - \frac{1}{2} \times 4 = 5$ ✓. Each law of logarithms corresponds to a law of indices: a product becomes a sum, a quotient a difference and a power a multiple.

Worked example 11: Changing the base

Use the change of base formula to evaluate $\log_7 25$ to 3 significant figures and to show that $\log_8 2 = \frac{1}{3}$.

$\log_a b = \dfrac{\log_c b}{\log_c a}$ for any base c. So $\log_7 25 = \dfrac{\ln 25}{\ln 7} = \dfrac{3.2189}{1.9459} = 1.65$. Also $\log_8 2 = \dfrac{\log_2 2}{\log_2 8} = \dfrac{1}{3}$. This is why scientific calculators only need $\log_{10}$ and $\ln$ keys. Check: $7^{1.654} = 25.0$.

Worked example 12: A model fitted by taking logarithms

The value of a collectable stamp was £12 in 2010 and £34 in 2020. Assuming exponential growth $V = ab^t$ (t years from 2010), find a and b and predict the value in 2030.

$a = 12$ since $V(0) = 12$. $V(10) = 34 = 12b^{10}$, so $b^{10} = \frac{34}{12} = 2.8333$ and $b = 2.8333^{0.1} = 1.1098$: the annual growth is about 11%. In 2030 $t = 20$ and $V = 12 \times 1.1098^{20} = 12 \times 8.027 = £96.3$. Equivalently $V = 12 \times \left(\frac{34}{12}\right)^2 = 12 \times 8.03 = 96.3$. Taking logs, $\log V = \log 12 + t \log b$ is a straight line with gradient $0.1 \times \log \frac{34}{12} = 0.0452$ per year.

Worked example 13: A trap in solving logarithmic equations

Solve $\log_3(x - 4) + \log_3(x + 4) = 2$ and explain which solution must be rejected.

$\log_3 [(x - 4)(x + 4)] = 2$ gives $x^2 - 16 = 9$, so $x^2 = 25$ and $x = \pm 5$. For $x = -5$ the terms $x - 4 = -9$ and $x + 4 = -1$ are negative, and the logarithm of a negative number is undefined, so $x = -5$ is rejected. The only solution is $x = 5$. Check: $\log_3 1 + \log_3 9 = 0 + 2 = 2$ ✓. Always check every solution in the original equation, because combining logarithms can introduce solutions that do not belong.

Worked example 14: Exponential equations that need a substitution and logs

Solve $e^{2x} - 4e^x - 5 = 0$ and explain why one root is rejected.

Let $y = e^x$: $y^2 - 4y - 5 = 0$, $(y-5)(y+1) = 0$. $e^x = 5$ gives $x = \ln 5 = 1.61$. $e^x = -1$ has no solution, because e^x is always positive. So the only solution is $x = \ln 5$. Check: $e^{2\ln 5} - 4 \times 5 - 5 = 25 - 20 - 5 = 0$ ✓. The graph of $e^{2x} - 4e^x - 5$ crosses the x-axis only once, at $x \approx 1.61$.

Worked example 15: Reading growth and decay from an equation

Compare $A = 300e^{0.04t}$ and $B = 500e^{-0.03t}$ and find when $A = B$.

A is growing at a rate of about 4% per unit time and B is decaying at about 3%. They are equal when $300e^{0.04t} = 500e^{-0.03t}$, so $e^{0.07t} = \frac{5}{3}$ and $t = \dfrac{\ln(5/3)}{0.07} = \dfrac{0.5108}{0.07} = 7.30$. At that time both equal $300e^{0.292} = 401.9$. Since A grows and B falls, A stays larger afterwards.

Worked example 16: Percentage growth rates and the number e

A sum of money grows by 5% per year. Write the model in the form $V = V_0 e^{kt}$ and find the time to triple.

Growth of 5% a year gives $V = V_0 \times 1.05^t$. To write this with base e note that $1.05 = e^{\ln 1.05}$, so $V = V_0 e^{0.04879t}$ and $k = \ln 1.05 = 0.0488$. Tripling needs $e^{kt} = 3$, so $t = \dfrac{\ln 3}{0.04879} = 22.5$ years. Using $1.05^t = 3$ and $t = \dfrac{\log 3}{\log 1.05}$ gives exactly the same answer, which shows that the two forms of the model are equivalent. A continuous growth rate k is always slightly smaller than the matching annual percentage rate, because compounding happens all the time.

Worked example 17: Finding the intersection of an exponential and a straight line numerically

Show that the equation $2^x = x + 3$ has a root between 2 and 3 and use a search to find it to 1 decimal place.

Let $f(x) = 2^x - x - 3$. Then $f(2) = 4 - 2 - 3 = -1$ and $f(3) = 8 - 3 - 3 = 2 > 0$, so the sign changes and there is a root in between. Try $f(2.5) = 5.657 - 5.5 = 0.157 > 0$ and $f(2.4) = 5.278 - 5.4 = -0.122$, so the root is between 2.4 and 2.5. $f(2.45) = 5.464 - 5.45 = 0.014 > 0$ so the root is below 2.45, meaning it is 2.4 to 1 decimal place (about 2.446). Equations mixing exponentials and linear terms usually cannot be solved with logarithms alone, so a numerical search or a graph is needed.

Note: choosing which base to use

Any base can be used to solve $a^x = b$, because the ratio $\dfrac{\log b}{\log a}$ does not depend on the base chosen. In pure mathematics the natural logarithm $\ln$ is preferred because the functions e^x and $\ln x$ have simple calculus. In science and engineering base 10 is common for orders of magnitude (pH, decibels, the Richter scale) and base 2 for information and computing. When you answer a question, use the base that makes the working shortest, but always give the final answer in the form requested, usually a decimal to 3 significant figures or an exact expression such as $\ln 5$.

Method: solving exponential and logarithmic equations

1. For an equation with the unknown in a power, isolate the power and take logarithms of both sides (or use the same base on both sides).
2. If it looks like a quadratic in a^x, substitute $y = a^x$, solve, then solve $a^x = y$ (rejecting any $y \leq 0$).
3. For logarithmic equations, combine the logs into one using the laws, then convert to an index equation.
4. Check that every solution makes each logarithm's argument positive.

Watch out: common logarithm errors

- Writing $\log(a + b) = \log a + \log b$. The law is for a product: $\log(ab) = \log a + \log b$.
- Writing $\dfrac{\log a}{\log b} = \log a - \log b$. They are different; the quotient is $\log_b a$.
- Forgetting to reject solutions that make the argument of a logarithm negative or zero.
- Rounding too early when using logs to solve; keep the full calculator display until the end.
- Confusing the growth rate in e^{kt} with a percentage: $k = 0.04$ is about a 4.08% rise per unit time.

Note: why logarithms linearise data

If $y = ab^x$ then $\log y = \log a + x \log b$, which is the equation of a straight line in the variables x and $\log y$. If $y = ax^n$ then $\log y = \log a + n \log x$, a straight line in $\log x$ and $\log y$. Scientists use this trick to decide which model fits the data: if a plot of $\log y$ against x is straight the growth is exponential, and if a plot of $\log y$ against $\log x$ is straight there is a power law. The gradient and intercept of the line then give the constants b, a and n.

Did you know?

John Napier, a Scottish landowner, published the first table of logarithms in 1614 to save astronomers months of multiplication, and the slide rule built on his idea was used by engineers until the 1970s. The number $e = 2.71828\ldots$ appears when interest

is compounded continuously: if £1 earns 100% interest per year, compounded more and more often, the amount tends to e pounds. It is also the natural base for growth and decay in biology, physics and finance.

Otto's tip

"Log" turns multiplying into adding, powers into multiplying, and a power equation into a linear equation.