Worksheet - Exponentials and logarithms

Year: Year 12 · 20

Show your working.

Question 1

Question 1 – Solving simple exponential equations *(Section A · Skills practice)*

Solve each equation. Give non-exact answers to 3 significant figures.

a. $2^x = 32$ [1 mark]
b. $3^{2x-1} = 81$ [3 marks]
c. $5^x = 20$ [3 marks]
d. $2^{3x-1} = 7$ [4 marks]

Question 2

Question 2 – The laws of logarithms *(Section A · Skills practice)*

Use the laws of logarithms.

a. Evaluate $\log_2 40 - \log_2 5$. [2 marks]
b. Evaluate $2\log_3 6 - \log_3 4$. [3 marks]
c. Evaluate $\log_5 \dfrac{1}{25}$ and $\log_{16} 4$. [3 marks]
d. Write $3\log a + \frac{1}{2}\log b - \log c$ as a single logarithm. [3 marks]

Question 3

Question 3 – Solving logarithmic equations *(Section A · Skills practice)*

Solve each equation.

a. $\log_3(x+2) = 2$ [2 marks]
b. $\log_2 x + \log_2(x-2) = 3$ [5 marks]
c. $\log_{10}(3x+1) - \log_{10}(x-1) = 1$ [4 marks]
d. $\ln(2x-1) = 3$, giving the answer to 3 significant figures [3 marks]

Question 4

Question 4 – Equations with e *(Section A · Skills practice)*

Solve each equation, giving exact answers in terms of natural logarithms where appropriate.

a. $e^{2x} = 12$ [2 marks]

b. $e^{2x} - 5e^{x} + 6 = 0$ [4 marks]
c. $3e^{-x} = 7$ [3 marks]

Question 5

Question 5 – Straight-line graphs from curved data *(Section A · Skills practice)*

Two experiments produced straight-line graphs after taking logarithms (base 10).

a. A graph of $\log_{10} y$ against x has gradient 0.3 and intercept 0.7. Find a and b in $y = ab^{x}$. [4 marks]
b. A graph of $\log_{10} y$ against $\log_{10} x$ has gradient 2.5 and intercept 0.6. Find a and n in $y = ax^{n}$. [4 marks]

Question 6

Question 6 – Sketching exponentials and logarithms *(Section A · Skills practice)*

Consider the functions $f(x) = 2^{x}$ and $g(x) = \log_2 x$.

a. State the coordinates where each graph crosses an axis. [2 marks]
b. State the equation of the asymptote of each graph. [2 marks]
c. Explain why the graphs are reflections of each other in the line $y = x$. [2 marks]
d. Solve $g(x) = 3$ and $f(x) = 3$. [3 marks]

Question 7

Question 7 – Doubling your savings *(Section B · Real-life practice)*

Jamal invests £5 000 at 4% compound interest per year. The value after n years is $V = 5000 \times 1.04^{n}$.

a. Find the value after 10 years. [2 marks]
b. How many whole years until the investment has doubled? [4 marks]
c. What annual rate of interest would be needed to reach £8 000 in 10 years? [4 marks]
d. Explain why logarithms are needed for part (b). [1 mark]

Question 8

Question 8 – Growing bacteria *(Section B · Real-life practice)*

A culture of bacteria grows according to $N = 200e^{0.3t}$, where t is the time in hours.

a. Find the number of bacteria at the start and after 8 hours (to the nearest whole number). [3 marks]
b. Find the doubling time in hours and minutes. [3 marks]
c. When will the culture reach 5 000 bacteria? [3 marks]
d. State one reason why the model cannot be accurate for large t. [1 mark]

Question 9

Question 9 – Carbon dating *(Section B · Real-life practice)*

Carbon-14 decays so that the amount remaining after t years is $N = N_0 e^{-kt}$. Its half-life is 5 730 years.

a. Show that $k = \dfrac{\ln 2}{5730}$ and calculate k to 3 significant figures. [3 marks]
b. What percentage remains after 2 000 years? [2 marks]
c. A wooden bowl has 30% of its original carbon-14. Estimate its age. [3 marks]
d. Explain why a 40 000-year-old sample is hard to date. [1 mark]

Question 10

Question 10 – Acidity and pH *(Section B · Real-life practice)*

The pH of a solution is $\text{pH} = -\log_{10}[H^+]$ where $[H^+]$ is the hydrogen ion concentration in mol/litre.

a. Find the pH when $[H^+] = 3.2 \times 10^{-5}$. [2 marks]
b. Find $[H^+]$ for blood with pH 7.4. [2 marks]
c. How many times more acidic (greater $[H^+]$) is a solution of pH 3 than a solution of pH 5? [3 marks]
d. Why does a drop of 1 in pH correspond to multiplying $[H^+]$ by 10? [2 marks]

Question 11

Question 11 – Measuring noise *(Section B · Real-life practice)*

Sound level in decibels is $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$ where I is the intensity in W/m^2 and $I_0 = 10^{-12}$.

a. A road has intensity 10^{-5} W/m^2. Find its sound level. [2 marks]
b. By how many decibels does the level rise if the intensity doubles? [3 marks]
c. A level of 85 dB is a legal limit at work. Find the intensity. [3 marks]
d. Explain why ten identical machines are only 10 dB louder than one. [2 marks]

Question 12

Question 12 – Fitting an exponential model *(Section B · Real-life practice)*

The number of followers of a new social media account (in hundreds) is recorded each week. The data are in the table, with the base-10 logarithm of each value.

Week x	0	1	2	3	4
Followers y	3.00	4.80	7.68	12.29	19.66

Week x	0	1	2	3	4
$\log_{10} y$	0.477	0.681	0.885	1.090	1.294

a. Show that the points $(x, \log_{10} y)$ lie close to a straight line and find its gradient. [3 marks]
b. Hence find the constants a and b in the model $y = ab^x$. [4 marks]
c. Predict the number of followers in week 10. [2 marks]
d. In which week does the model predict 100 000 followers (1 000 hundred)? [3 marks]

Question 13

Question 13 – Metabolic rate *(Section B · Real-life practice)*

Biologists suggest that the daily energy use R (kcal) of a mammal of mass m kg follows $R = am^n$. The table shows data for three animals.

Mass m (kg)	1	16	81
R (kcal)	70	560	1890

a. Take logarithms of the relationship to show that $\log R$ against $\log m$ is a straight line. [2 marks]
b. Use the first two points to find n and a. [4 marks]
c. Check that the third point fits. [2 marks]
d. Predict R for a 625 kg horse. [2 marks]

Question 14

Question 14 – Car depreciation *(Section B · Real-life practice)*

A new car costs £18 000. Its value after t years is modelled by $V = 18000 \times 0.82^t$.

a. What percentage of its value does the car lose each year? [1 mark]
b. Find the value after 3 years to the nearest £10. [2 marks]
c. After how many years will the car be worth less than £5 000? [4 marks]
d. How much does it lose in the first year? [1 mark]

Question 15

Question 15 – Caffeine in the body *(Section C · Problem solving)*

The half-life of caffeine in the body is 5 hours. After drinking a 200 mg dose, the amount in the body is $M = 200 \times \left(\frac{1}{2}\right)^{t/5}$ mg, where t is in hours.

t (h)	0	5	10	15
M (mg)	200	100	50	25

a. Write M in the form $200e^{-kt}$, giving k to 3 significant figures. [3 marks]
b. How long until fewer than 10 mg remain? [4 marks]
c. At $t = 5$ a second 100 mg dose is taken. Find the total amount in the body at $t = 8$. [4 marks]
d. Explain why the total at $t = 8$ is almost the same from each dose. [1 mark]

Question 16

Question 16 – Logarithmic simultaneous equations *(Section C · Problem solving)*

Solve each problem.

a. $\log_2 x + \log_2 y = 5$ and $x - y = 4$, given $x, y > 0$. [6 marks]
b. Solve $\log_2 x = \log_x 2$ (with $x > 0$, $x \neq 1$). [4 marks]
c. Prove that $\log_a b \times \log_b a = 1$. [3 marks]

Question 17

Question 17 – Fitting a growth model *(Section C · Problem solving)*

A town\'s population (in people) is recorded at two-year intervals. The model $P = 1200e^{kt}$ is proposed, with t in years.

t	0	2	4	6
P	1200	1466	1790	2187
$\ln P$	7.090	7.290	7.490	7.690

a. Show that the values of $\ln P$ increase by a constant amount per 2 years and find k. [3 marks]
b. Find the time for the population to double. [3 marks]
c. Predict the population at $t = 10$ to the nearest 100. [2 marks]
d. The council plans for 5 000 people. When is this reached? [3 marks]

Question 18

Question 18 – How long a medicine lasts *(Section C · Problem solving)*

A patient takes a 150 mg tablet every morning. During each day the body keeps 70% of the drug. Immediately after the nth tablet the amount in the body is $S_n = 150 + 150(0.7) + \cdots + 150(0.7)^{n-1}$.

Tablet	1	2	3	4
Amount (mg)	150	255	328.5	379.95

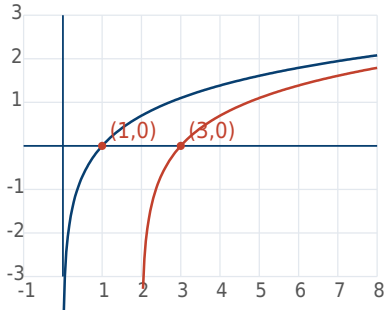
a. Check the amount after tablet 3 and identify the type of series. [2 marks]
b. Show that $S_n = 500(1 - 0.7^n)$. [2 marks]
c. What is the long-term maximum amount in the body? [2 marks]

d. After how many tablets is the amount within 1% of the maximum? [4 marks]

Question 19

Question 19 – Graphs of logarithms and their transformations *(Section C · Problem solving)*

The graph shows $y = \ln x$ (blue) and $y = \ln(x - 2)$ (red).



a. Describe the transformation from $y = \ln x$ to $y = \ln(x - 2)$ and state the equation of the asymptote of each graph. [3 marks]
b. Find the exact x-coordinate where $y = \ln(x - 2)$ equals 2. [2 marks]
c. $f(x) = 3^x + 1$. Find the inverse function $f^{-1}(x)$ and its domain. [4 marks]
d. Solve $\ln x - \ln(x - 2) = \ln 3$. [4 marks]

Question 20

Question 20 – Challenge: three equations *(Section D · Challenge)*

Solve each equation exactly.

a. $3^{2x} - 10 \times 3^x + 9 = 0$ [4 marks]
b. $\log_2(x - 1) + \log_2(x + 2) = 2$ [4 marks]
c. $2^x = 3^{x-1}$, giving the answer to 3 significant figures [4 marks]