Answers - Exponentials and logarithms

Year: Year 12

Answer to Question 1

a. Write 32 as a power of 2 and compare the indices. $32 = 2^5$ so $x = \mathbf{5}$. Check: $2^5 = 32$ ✓.

b. Write both sides as powers of 3, then equate the indices. $81 = 3^4$, so $2x - 1 = 4$ and $x = \mathbf{2.5}$. Check: $3^4 = 81$ ✓.

c. 20 is not an exact power of 5, so take logarithms of both sides. $x = \log_5 20 = \dfrac{\log 20}{\log 5} = \mathbf{1.86}$. Check: $5^{1.86} \approx 20$ ✓.

d. Use $\log a^n = n \log a$ to bring the power down. Take logs: $(3x - 1)\log 2 = \log 7$, so $3x - 1 = \dfrac{\log 7}{\log 2} = 2.807$ and $x = \dfrac{3.807}{3} = \mathbf{1.27}$. So $x = 1.27$ to 3 significant figures.

Answer to Question 2

a. Use $\log_a m - \log_a n = \log_a \dfrac{m}{n}$. $\log_2 \dfrac{40}{5} = \log_2 8 = \mathbf{3}$. So the value is 3, because $2^3 = 8$.

b. First use $k \log_a m = \log_a m^k$, then the subtraction law. $\log_3 6^2 - \log_3 4 = \log_3 \dfrac{36}{4} = \log_3 9 = \mathbf{2}$. So the value is 2.

c. Write each number as a power of the base. $\log_5 \dfrac{1}{25} = \log_5 5^{-2} = -2$. $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$. So $\log_5 \frac{1}{25} = -2$ and $\log_{16} 4 = \frac{1}{2}$.

d. Use the power law on each term: $3 \log a = \log a^3$ and $\frac{1}{2} \log b = \log \sqrt{b}$; then combine with the multiplication and division laws. $\log \dfrac{a^3 \sqrt{b}}{c}$. So the single logarithm is $\log \dfrac{a^3 \sqrt{b}}{c}$.

Answer to Question 3

a. Convert to exponential form: $\log_a x = n \iff x = a^n$. $x + 2 = 3^2 = 9$, so $x = \mathbf{7}$. Check: $\log_3 9 = 2$ ✓.

b. Combine the logarithms using the multiplication law, then convert to exponential form. $\log_2 x(x - 2) = 3$, so $x(x - 2) = 8$, $x^2 - 2x - 8 = 0$, $(x - 4)(x + 2) = 0$.

$x = -2$ is rejected (log of a negative). $x =$ **4**. Check: $\log_2 4 + \log_2 2 = 2 + 1 = 3$ ✓.

c. Use the division law, then $\log_{10} N = 1 \iff N = 10$. $\log_{10} \dfrac{3x+1}{x-1} = 1$, so $\dfrac{3x+1}{x-1} = 10$, $3x + 1 = 10x - 10$, $7x = 11$, $x = \dfrac{11}{7}$. Check $x - 1 = \frac{4}{7} > 0$ ✓. So $x = \dfrac{11}{7}$.

d. The inverse of $\ln$ is e^x, so apply the exponential to both sides. $2x - 1 = e^3 = 20.09$, so $x = \dfrac{21.09}{2} =$ **10.5**. So $x = 10.5$ to 3 significant figures.

Answer to Question 4

a. Take natural logarithms of both sides: $\ln e^{2x} = 2x$. $2x = \ln 12$, so $x = \frac{1}{2}\ln 12 =$ **1.24**. The exact answer is $x = \frac{1}{2}\ln 12$.

b. Notice that $e^{2x} = (e^x)^2$, so this is a quadratic in e^x. Let $y = e^x$: $y^2 - 5y + 6 = 0$, $(y-2)(y-3) = 0$, so $e^x = 2$ or 3 and $x = \ln 2$ or $\ln 3$. Both solutions are valid because e^x can equal any positive number.

c. Divide by 3 first, then take natural logarithms. $e^{-x} = \frac{7}{3}$, $-x = \ln \frac{7}{3}$, so $x = -\ln \frac{7}{3} = \ln \frac{3}{7} = -0.847$. So $x = \ln \frac{3}{7} \approx -0.847$.

Answer to Question 5

a. Take logarithms of $y = ab^x$ so that it has the form $Y = c + mX$ with $Y = \log y$. $y = ab^x \Rightarrow \log y = \log a + x \log b$. Intercept $\log a = 0.7$ so $a = 10^{0.7} =$ **5.01**; gradient $\log b = 0.3$ so $b = 10^{0.3} =$ **1.995**. So the model is $y = 5.01 \times 1.995^x$, roughly $y = 5 \times 2^x$.

b. Take logarithms of $y = ax^n$; this time both axes are logarithmic. $y = ax^n \Rightarrow \log y = \log a + n \log x$. Gradient $n =$ **2.5**; intercept $\log a = 0.6$, so $a = 10^{0.6} =$ **3.98**. So the model is $y = 3.98x^{2.5}$.

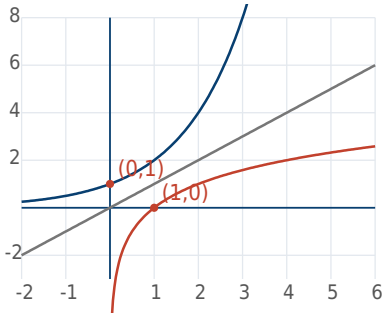
Answer to Question 6

a. Use $2^0 = 1$ and $\log_2 1 = 0$. f crosses the y-axis at $(0, 1)$. g crosses the x-axis at $(1, 0)$.

b. 2^x is always positive and tends to 0 as $x \to -\infty$; $\log_2 x$ is only defined for $x > 0$. f : the x-axis, $y = 0$. g: the y-axis, $x = 0$.

c. g is the inverse of f: $y = 2^x \iff x = \log_2 y$. Swapping x and y reflects the graph in the line $y = x$. So every point (a, b) on one graph corresponds to the point (b, a) on the other.

d. Rewrite each equation in the other form. $g(x) = 3$: $x = 2^3 =$ **8**. $f(x) = 3$: $x = \log_2 3 =$ **1.58**. So $x = 8$ and $x = \log_2 3 \approx 1.58$ respectively.



Answer to Question 7

a. Substitute $n = 10$ into the formula. $5000 \times 1.04^{10} =$ **£7 401.22**. After 10 years the investment is worth £7 401.22.
b. Solve $5000 \times 1.04^n = 10000$ by dividing by 5000 and taking logarithms. $1.04^n = 2$, so $n = \dfrac{\log 2}{\log 1.04} = 17.67$. After **18 whole years**. The investment has doubled after 18 whole years.
c. Write the multiplier as $1 + r$ and take the tenth root. $5000(1+r)^{10} = 8000$, $(1+r)^{10} = 1.6$, $1 + r = 1.6^{0.1} = 1.0481$. The rate is **4.81%**. An annual rate of about 4.81% is needed.
d. The unknown n is in the power, and logarithms undo powers. Without logarithms we could only find n by trial and improvement.

Answer to Question 8

a. Substitute $t = 0$ and $t = 8$; note that $e^0 = 1$. $t = 0$: **200**. $t = 8$: $200e^{2.4} =$ **2 205**. There are 200 bacteria at the start and about 2 205 after 8 hours.
b. Doubling means $N = 400$, so solve $200e^{0.3t} = 400$. $e^{0.3t} = 2$, $t = \dfrac{\ln 2}{0.3} = 2.31$ h = **2 h 19 min**. 0.31 h is about 19 minutes, so the doubling time is 2 h 19 min.
c. Solve $200e^{0.3t} = 5000$ by dividing by 200 and taking natural logarithms. $e^{0.3t} = 25$, $t = \dfrac{\ln 25}{0.3} =$ **10.7 h**. The culture reaches 5 000 bacteria after about 10.7 hours.
d. The food supply and space would run out, so growth cannot continue exponentially forever. In practice the growth slows down and levels off.

Answer to Question 9

a. The half-life is the time for the amount to halve; take logarithms of $e^{-5730k} = \frac{1}{2}$. After 5 730 years $N = \frac{1}{2}N_0$, so $e^{-5730k} = \frac{1}{2}$ and $k = \dfrac{\ln 2}{5730} = 1.21 \times 10^{-4}$. So $k = 1.21 \times 10^{-4}$ per year.

b. Substitute $t = 2000$ into $\frac{N}{N_0} = e^{-kt}$. $e^{-0.1209\ldots\times 2}$: $e^{-0.2419} = 0.785$, so **78.5%**. About 78.5% of the carbon-14 remains.
c. Solve $e^{-kt} = 0.3$ by taking natural logarithms. $e^{-kt} = 0.3$, $t = \frac{-\ln 0.3}{k} = \frac{1.204}{1.2097 \times 10^{-4}} =$ **9 950 years** (to 3 s.f.). The bowl is about 9 950 years old.
d. After 40 000 years only $e^{-4.84} \approx 0.8\%$ remains, which is too little to measure accurately. Small measurement errors then give large errors in the age.

Answer to Question 10

a. Substitute into the formula and use a calculator. $-\log_{10}(3.2 \times 10^{-5}) = 4.49$. The pH is 4.49, so the solution is acidic.
b. Rearrange: $[H^+] = 10^{-\text{pH}}$. $[H^+] = 10^{-7.4} = 3.98 \times 10^{-8}$ mol/litre.
c. Convert each pH to a concentration and divide. $[H^+]$ for pH 3 is 10^{-3} and for pH 5 is 10^{-5}. The ratio is $10^2 =$ **100 times**. The pH 3 solution is 100 times more acidic.
d. $[H^+] = 10^{-\text{pH}}$, so decreasing the pH by 1 multiplies by $10^1 = 10$. This is why the pH scale is called a logarithmic scale.

Answer to Question 11

a. Divide the intensity by I_0: $\frac{10^{-5}}{10^{-12}} = 10^7$. $10\log_{10}(10^7) =$ **70 dB**. The road has a sound level of 70 dB.
b. Use the multiplication law of logarithms to separate the factor 2. New level $= 10\log\frac{2I}{I_0} = 10\log 2 + 10\log\frac{I}{I_0}$, so it rises by $10\log 2 =$ **3.01 dB**. Doubling the intensity always adds about 3 dB.
c. Divide by 10 and convert from logarithmic form. $85 = 10\log\frac{I}{I_0}$ gives $\frac{I}{I_0} = 10^{8.5}$, $I = 10^{8.5} \times 10^{-12} = 3.16 \times 10^{-4}$ **W/m^2**. The intensity is about 3.16×10^{-4} W/m^2.
d. Ten machines multiply the intensity by 10, adding $10\log_{10} 10 = 10$ dB. The decibel scale grows with the logarithm of the intensity, not the intensity itself.

Answer to Question 12

a. Look at the differences between consecutive values of $\log_{10} y$. The differences in $\log_{10} y$ are 0.204, 0.204, 0.205, 0.204: constant, so the points lie on a line. Gradient $\approx \frac{1.294 - 0.477}{4} = 0.204$. So the points lie very close to a straight line with gradient 0.204.

b. Compare $\log y = \log a + x \log b$ with the line $Y = 0.477 + 0.204x$. $\log y = \log a + x \log b$. Intercept $0.477 = \log a$, so $a = 3.0$; gradient $0.204 = \log b$, so $b = 10^{0.204} = 1.60$. Model: $y = 3 \times 1.6^x$.

c. Substitute $x = 10$ into the model. $y = 3 \times 1.6^{10} =$ **329.9 hundred = 33 000 followers**. The model predicts about 33 000 followers in week 10.

d. Solve $3 \times 1.6^x = 1000$ by dividing by 3 and taking logarithms. $3 \times 1.6^x = 1000$: $x = \dfrac{\log(333.3)}{\log 1.6} = 12.35$. In **week 13**. Because the week must be a whole number, round up to week 13.

Answer to Question 13

a. Take logarithms of both sides and use the multiplication and power laws. $\log R = \log a + n \log m$: a straight line with gradient n and intercept $\log a$. So the points $(\log m, \log R)$ lie on a straight line.

b. Substitute each data point into $R = am^n$. $m = 1$: $R = a = 70$. $m = 16$: $560 = 70 \times 16^n$, $16^n = 8$, $n = \dfrac{\log 8}{\log 16} = \dfrac{3}{4}$. So $R = 70m^{0.75}$.

c. Substitute $m = 81$ into the model; $81^{0.75} = (\sqrt[4]{81})^3 = 3^3$. $m = 81$: $81^{0.75} = 27$ and $70 \times 27 = 1890$ ✓. The third point fits the model exactly.

d. $625^{0.75} = (\sqrt[4]{625})^3 = 5^3 = 125$. $625^{0.75} = 125$, $R = 70 \times 125 =$ **8 750 kcal**. The model predicts that the horse uses about 8 750 kcal per day.

Answer to Question 14

a. The multiplier 0.82 means the car keeps 82% of its value each year. $1 - 0.82 =$ **18%** a year.

b. Substitute $t = 3$. $18000 \times 0.82^3 =$ **£9 920**. After 3 years the car is worth about £9 920.

c. Solve 18000×0.82^t; dividing by $\ln 0.82$ reverses the inequality. 0.82^t, $t > \dfrac{\ln 0.2778}{\ln 0.82} = 6.45$. After **6.5 years** (during the 7th year). So the car is worth less than £5 000 after 7 whole years.

d. The loss in the first year is 18% of the original price. $18000 \times 0.18 =$ **£3 240**.

Answer to Question 15

a. Use $\frac{1}{2} = e^{-\ln 2}$, so $\left(\frac{1}{2}\right)^{t/5} = e^{-(\ln 2)t/5}$. $\left(\frac{1}{2}\right)^{t/5} = e^{-(\ln 2/5)t}$, so $k = \dfrac{\ln 2}{5} =$ **0.139**. So $M = 200e^{-0.139t}$.

b. Solve M using logarithms. $200 \times 2^{-t/5}$ gives $2^{t/5} > 20$, $\dfrac{t}{5} > \dfrac{\ln 20}{\ln 2} = 4.32$, $t >$

21.6 hours. Less than 10 mg remains after about 21.6 hours.

c. Treat the two doses separately: at $t = 8$ the first dose has decayed for 8 hours and the second for 3 hours. First dose: $200 \times 2^{-8/5} = 65.98$ mg. Second dose (3 hours old): $100 \times 2^{-3/5} = 65.98$ mg. Total = **132.0 mg**. So the total amount at $t = 8$ is about 132 mg.

d. $200 \times 2^{-1.6}$ and $100 \times 2^{-0.6}$ are equal because $200 \times 2^{-1} = 100$. So twice the dose taken 5 hours earlier (one half-life) leaves exactly the same amount as the smaller later dose.

Answer to Question 16

a. Combine the logarithms to get $xy = 32$, then substitute $x = y + 4$. $\log_2(xy) = 5$, so $xy = 32$. With $x = y + 4$: $y^2 + 4y - 32 = 0$, $(y + 8)(y - 4) = 0$, $y = 4$ (positive). So $x = 8$**,** $y = 4$. Check: $\log_2 8 + \log_2 4 = 3 + 2 = 5$ ✓. So $x = 8$ and $y = 4$.

b. Use the change of base rule $\log_x 2 = \dfrac{1}{\log_2 x}$. Let $u = \log_2 x$. Then $\log_x 2 = \dfrac{1}{u}$, so $u = \dfrac{1}{u}$, $u^2 = 1$, $u = \pm 1$. So $x = 2$ or $x = \frac{1}{2}$. Both values are valid because they are positive and not equal to 1.

c. Write each logarithm in exponential form. Let $\log_a b = p$, so $a^p = b$. Then $\log_b a = q$ means $b^q = a$. Substitute: $(a^p)^q = a$, so $a^{pq} = a^1$, hence $pq = 1$. So $\log_a b \times \log_b a = 1$, as required.

Answer to Question 17

a. Taking natural logarithms gives a straight line in t. $\ln P = \ln 1200 + kt$. The increases are 0.200 each time, so $k = \dfrac{0.200}{2} =$ **0.1**. So $\ln P$ increases at a constant rate and $k = 0.1$.

b. Doubling means $e^{0.1t} = 2$. $e^{0.1t} = 2$, $t = \dfrac{\ln 2}{0.1} =$ **6.93 years**. The population doubles in about 6.93 years.

c. Substitute $t = 10$ into the model. $1200e^1 = 3262 \approx$ **3 300**. The predicted population is about 3 300.

d. Solve $1200e^{0.1t} = 5000$ with natural logarithms. $e^{0.1t} = \dfrac{5000}{1200} = 4.1667$, $t = 10 \ln 4.1667 =$ **14.3 years**. The population reaches 5 000 after about 14.3 years.

Answer to Question 18

a. Add the first three terms; each term is 0.7 times the previous one. $150 + 105 + 73.5 = 328.5$ ✓. A geometric series with $a = 150$, $r = 0.7$.

b. Use $S_n = \frac{a(1-r^n)}{1-r}$. $S_n = \frac{150(1-0.7^n)}{1-0.7} = 500(1-0.7^n)$. This is the required formula.
c. Since $|r|$, $0.7^n \to 0$ and the sum tends to S_∞. $S_\infty = \frac{150}{0.3} =$ **500 mg**. The amount in the body never exceeds 500 mg.
d. 1% of 500 is 5, so we need $S_n > 495$. $500(1-0.7^n) > 495$ gives 0.7^n, $n > \frac{\ln 0.01}{\ln 0.7} = 12.9$. After **13 tablets**. After 13 tablets the amount is within 1% of the maximum.

Answer to Question 19

a. Replacing x by $x-2$ moves the graph 2 units to the right. A translation by $\begin{pmatrix}2\\0\end{pmatrix}$. Asymptotes: $x=0$ for $\ln x$ and $x=2$ for $\ln(x-2)$.
b. Convert from logarithmic form using e. $x-2=e^2$, so $x=2+e^2=9.39$. The exact answer is $x=2+e^2$.
c. Write $y=f(x)$, make x the subject, then swap the letters. $y=3^x+1 \Rightarrow 3^x=y-1 \Rightarrow x=\log_3(y-1)$. So $f^{-1}(x)=\log_3(x-1)$, domain $x>1$ (the range of f).
d. Use the division law to combine the logarithms on the left. $\ln\frac{x}{x-2}=\ln 3$, so $x=3(x-2)$, $2x=6$, $x=$ **3**. Check: $\ln 3-\ln 1=\ln 3$ ✓. So $x=3$.

Answer to Question 20

a. Notice that $3^{2x}=(3^x)^2$, so substitute $y=3^x$ to get a quadratic. Let $y=3^x$: $y^2-10y+9=0$, $(y-1)(y-9)=0$. $3^x=1$ gives $x=0$, $3^x=9$ gives $x=2$. $x=0$ **or** 2. Check: $81-90+9=0$ ✓.
b. Combine the logarithms with the multiplication law and convert to exponential form. $(x-1)(x+2)=2^2=4$, so $x^2+x-6=0$, $(x+3)(x-2)=0$. $x=-3$ is invalid because $x-1$. $x=2$. Check: $\log_2 1+\log_2 4=0+2=2$ ✓.
c. Take natural logarithms and collect the terms in x. Take logs: $x\ln 2=(x-1)\ln 3$, so $x(\ln 3-\ln 2)=\ln 3$ and $x=\frac{\ln 3}{\ln 3-\ln 2}=$ **2.71**. Check: $2^{2.71}\approx 6.54$ and $3^{1.71}\approx 6.54$ ✓.