Test answers - Differentiation

Year: Year 12 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Write every term as a power of x: $-\dfrac{5}{x} = -5x^{-1}$ and $\sqrt{x} = x^{1/2}$; in (b) divide each term of the numerator by x^2 first.

(a) $12x^2 + 5x^{-2} + \frac{1}{2}x^{-1/2}$ (2). (b) $y = x + 2x^{-1}$ so $\dfrac{dy}{dx} = 1 - 2x^{-2}$ (2).

So (a) $\dfrac{dy}{dx} = 12x^2 + \dfrac{5}{x^2} + \dfrac{1}{2\sqrt{x}}$ and (b) $\dfrac{dy}{dx} = 1 - \dfrac{2}{x^2}$.

A common mistake in (b) is to differentiate the numerator and the denominator separately; always divide by x^2 first.

In (a), the derivative of $-5x^{-1}$ is $+5x^{-2}$: the two minus signs combine, and this sign is where most marks are lost.

Answer to Question 2 (4 marks)

Differentiate to find the gradient at $x = 2$, then use $y - y_1 = m(x - x_1)$; the normal has gradient $-\dfrac{1}{m}$.

$\dfrac{dy}{dx} = 3x^2 - 4x = 4$ at $x = 2$ (1). Tangent: $y - 3 = 4(x - 2)$, $y = 4x - 5$ (1). Normal gradient $-\frac{1}{4}$ (1): $y - 3 = -\frac{1}{4}(x - 2)$, $x + 4y = 14$ (1).

So the tangent is $y = 4x - 5$ and the normal is $x + 4y = 14$.

Check: the point $(2, 3)$ lies on both lines, since $4 \times 2 - 5 = 3$ ✓ and $2 + 4 \times 3 = 14$ ✓.

The normal can also be written $y = -\frac{1}{4}x + \frac{7}{2}$; any equivalent form scores full marks.

Answer to Question 3 (4 marks)

Set $\dfrac{dy}{dx} = 0$, factorise, find the y-coordinates from the original equation and classify with the second derivative.

$y' = 6x^2 - 18x + 12 = 6(x - 1)(x - 2)$ (1), $x = 1$: $y = 5$; $x = 2$: $y = 4$ (1). $y'' = 12x - 18$: at $x = 1$, -6 maximum $(1, 5)$ (1); at $x = 2$, 6 minimum $(2, 4)$ (1).

So $(1, 5)$ is a maximum and $(2, 4)$ is a minimum.

Check with nearby values: $y(0.9) = 4.968$ and $y(1.1) = 4.972$ are both less than 5, so $(1, 5)$ is indeed a local maximum.

Answer to Question 4 (4 marks)

A function is increasing when its derivative is positive for every x.

$f' = 3x^2 + 3 > 0$ for all x (2). $g' = 3x^2 + k \geq 0$ for all x needs $k \geq 0$ (2).

Since $3x^2 \geq 0$ for all x, we need $k \geq 0$; if $k < 0$, $g'(0) = k < 0$ and g decreases near 0. So $k \geq 0$.

For example, $k = -3$ gives $g'(x) = 3(x^2 - 1)$, which is negative for $-1 < x < 1$, so that g is not increasing everywhere.

In the first part, a complete answer states that $3x^2 \geq 0$, so $3x^2 + 3 \geq 3 > 0$. For g, the value $k = 0$ is allowed: $g(x) = x^3$ has $g'(0) = 0$ but is still increasing everywhere.

Answer to Question 5 (4 marks)

a. The wall forms one side, so the fencing covers two widths and one length. Length $= 80 - 2x$ so $A = x(80 - 2x)$ (1) This is the required expression.
b. Differentiate and set the derivative equal to zero, then check with the second derivative. $A' = 80 - 4x = 0$, $x = 20$ (1), $A'' = -4 < 0$ (1), $A = 800$ m^2 (1) The enclosure should be 20 m wide and 40 m long, giving a maximum area of 800 m^2. Check: $20 \times 40 = 800$ and the fencing used is $20 + 40 + 20 = 80$ m ✓.

In (b), the second-derivative test earns the mark for justifying a maximum; showing that $A(19) = 798$ and $A(21) = 798$ are both below 800 is also accepted.