Lesson - Differentiation

Year: Year 12

Differentiation finds the rate at which one quantity changes with another. Geometrically, the derivative is the gradient of the tangent to a curve. It is used for tangents, normals, rates of change and optimisation.

Starter activity

A cyclist records the distance travelled each minute: 0, 150, 340, 540, 700, 800 metres. Work out the distance covered in each minute. When was the cyclist going fastest? Estimate the speed at exactly 3 minutes by using the distances at 2.5 and 3.5 minutes if you could measure them. Why does using smaller and smaller time intervals give a better estimate of the speed at a single instant? This idea is the heart of the derivative.

Gradient and first principles

Definition: derivative

The gradient of a chord from $(x, f(x))$ to $(x+h, f(x+h))$ is $\dfrac{f(x+h)-f(x)}{h}$. As $h \to 0$ the chord becomes the tangent, and the **derivative** is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

Worked example 1 (first principles)

Differentiate $f(x) = x^2$ from first principles.

$\dfrac{(x+h)^2 - x^2}{h} = \dfrac{2xh + h^2}{h} = 2x + h$. As $h \to 0$ this tends to $2x$, so $f'(x) = 2x$. The same method for x^3 uses $(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$ and gives $3x^2$.

Differentiating powers of x

Rule: differentiating ax^n

If $y = ax^n$ (any real n), then $\dfrac{dy}{dx} = anx^{n-1}$. A constant differentiates to 0, and you differentiate term by term. The second derivative $\dfrac{d^2y}{dx^2}$ is the derivative of the derivative; it describes how the gradient is changing.

Method: differentiating roots and reciprocals

1. Write roots and reciprocals as powers: $\sqrt{x} = x^{1/2}$, $\dfrac{1}{x^2} = x^{-2}$.
2. Expand any brackets first.
3. Differentiate each term: bring the power down, then take one off the power.

Worked example 2

Differentiate $y = 3\sqrt{x} - \dfrac{2}{x^2}$ and $y = (x+2)(x-3)$.

$y = 3x^{1/2} - 2x^{-2}$, so $\dfrac{dy}{dx} = \dfrac{3}{2}x^{-1/2} + 4x^{-3}$. Also $y = x^2 - x - 6$ gives $\dfrac{dy}{dx} = 2x - 1$.

Tangents and normals

Definition: tangent and normal

At $x = a$ the **tangent** has gradient $f'(a)$. The **normal** is perpendicular to the tangent, with gradient $-\dfrac{1}{f'(a)}$.

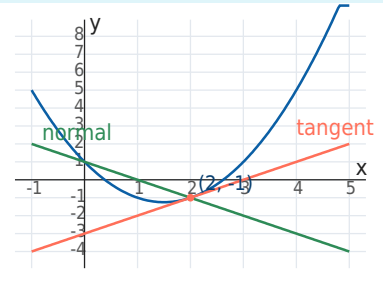
Method: tangent and normal at a point

1. Find y at the given x.
2. Differentiate and substitute x to get the gradient.
3. Use $y - y_1 = m(x - x_1)$ with m for the tangent, $-\dfrac{1}{m}$ for the normal.

Worked example 3

Find the tangent and normal to $y = x^2 - 3x + 1$ at $x = 2$.

$y = -1$. $\dfrac{dy}{dx} = 2x - 3 = 1$. Tangent: $y + 1 = x - 2 \Rightarrow y = x - 3$. Normal gradient -1: $y + 1 = -(x - 2) \Rightarrow y = -x + 1$.



Worked example 4

Find the tangent to $y = x^2 - 4x + 7$ that is parallel to $y = 2x + 1$.

Equal gradients, so $2x - 4 = 2$ and $x = 3$. Then $y = 9 - 12 + 7 = 4$, and the tangent is $y - 4 = 2(x - 3)$, that is $y = 2x - 2$.

Increasing and decreasing functions

Definition: increasing and decreasing

f is **increasing** where $f'(x) > 0$ and **decreasing** where $f'(x)$.

Worked example 5

Find where $y = x^3 - 3x^2 - 9x$ is increasing.

$3x^2 - 6x - 9 > 0 \Rightarrow 3(x - 3)(x + 1) > 0$ (sketch the $\cup$), so x or $x > 3$.

Worked example 6

Show that $y = x^3 + 3x + 5$ is increasing for all x.

$\frac{dy}{dx} = 3x^2 + 3 \geq 3 > 0$ for every real x, because $3x^2 \geq 0$.

Stationary points

Definition: stationary point

A **stationary point** is a point where $\frac{dy}{dx} = 0$.

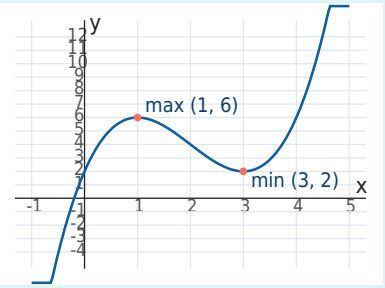
Rule: classifying stationary points

- $\frac{d^2y}{dx^2} > 0$ is a minimum, is a maximum. If it is 0 the test fails: check the sign of $\frac{dy}{dx}$ either side.
- Gradient sign either side: $+ \to -$ is a maximum, $- \to +$ is a minimum.

Worked example 7

Find and classify the stationary points of $y = x^3 - 6x^2 + 9x + 2$.

$\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, so $x = 1$ or 3. $\frac{d^2y}{dx^2} = 6x - 12$: at $x = 1$ it is -6, a maximum at $(1, 6)$; at $x = 3$ it is $6 > 0$, a minimum at $(3, 2)$.



Worked example 8

The curve $y = x^2 + px + q$ has a turning point at $(3, -4)$. Find p and q.

$\frac{dy}{dx} = 2x + p = 0$ at $x = 3$, so $p = -6$. Substituting the point: $9 - 18 + q = -4$, so $q = 5$.

Optimisation

Method: optimisation

1. Write the quantity to optimise as a function of *one* variable.
2. Differentiate and set the derivative to zero.
3. Prove it is a maximum or minimum (second derivative), in words.
4. Answer the question in context, with units, and check it is within the physical limits.

Worked example 9

A rectangle has perimeter 40 cm. Find the largest possible area.

Let the sides be x and $20 - x$. Area $A = 20x - x^2$, $\frac{dA}{dx} = 20 - 2x = 0$ gives $x = 10$; $\frac{d^2A}{dx^2} = -2$, a maximum. The largest area is 100 cm^2 (a square).

Watch out

Do not differentiate a product by multiplying the derivatives: expand first. Write $\frac{1}{x}$ as x^{-1} before differentiating. Do not use the tangent gradient for the normal. At a stationary point give the y-coordinate too, and prove the nature when asked.

Worked example 10: Differentiating with brackets and fractions

Differentiate $y = \dfrac{(x+1)^2}{\sqrt{x}}$.

Expand and divide term by term: $\dfrac{x^2+2x+1}{x^{1/2}} = x^{3/2} + 2x^{1/2} + x^{-1/2}$. Differentiate: $\dfrac{dy}{dx} = \frac{3}{2}x^{1/2} + x^{-1/2} - \frac{1}{2}x^{-3/2}$. At $x = 4$ this is $\frac{3}{2} \times 2 + \frac{1}{2} - \frac{1}{2} \times \frac{1}{8} = 3 + 0.5 - 0.0625 = 3.4375$. There is no product or quotient rule at this level: always rewrite the expression as a sum of powers before differentiating.

Worked example 11: Rate of change at a point

The depth of water in a tank is $d = 0.02t^3 - 0.3t^2 + 1.5t$ metres after t hours. Find the rate at which the depth is changing after 5 hours and the time when the depth stops changing.

$\dfrac{dd}{dt} = 0.06t^2 - 0.6t + 1.5$. At $t = 5$: $1.5 - 3 + 1.5 = 0$, so the depth is momentarily constant already at 5 hours. Solving $0.06t^2 - 0.6t + 1.5 = 0$ gives $0.06(t-5)^2 = 0$, so $t = 5$ is a repeated root: the rate never becomes negative, so $t = 5$ is a point of inflection with zero gradient. The depth at that time is $2.5 - 7.5 + 7.5 = 2.5$ m. The tank keeps filling afterwards.

Worked example 12: Finding the gradient from the tangent condition

The tangent to $y = x^2 + px + q$ at $x = 3$ is $y = 4x - 5$. Find p and q.

At $x = 3$ the tangent gives $y = 7$, so the curve passes through $(3, 7)$: $9 + 3p + q = 7$. The gradient is $2x + p = 6 + p$ and must equal 4, so $p = -2$. Then $9 - 6 + q = 7$ gives $q = 4$. So the curve is $y = x^2 - 2x + 4$. Check the tangent: gradient at $x = 3$ is $2 \times 3 - 2 = 4$ ✓ and the point $(3, 7)$ lies on both curve and line ✓.

Worked example 13: Interpreting the second derivative

For $y = x^3 - 6x^2$, find the point where the curve changes from concave to convex and describe its gradient there.

$y' = 3x^2 - 12x$ and $y'' = 6x - 12$. The second derivative is zero at $x = 2$, negative for x (concave, bending downwards) and positive for $x > 2$ (convex, bending upwards). So $(2, -16)$ is a point of inflection with gradient $y'(2) = 12 - 24 = -12$, the minimum gradient on the curve. Notice that the gradient is least where the second derivative changes sign, which is the same idea as the steepest part of the rollercoaster.

Worked example 14: Showing a function is always increasing or decreasing

Show that $f(x) = x^3 - 3x^2 + 3x + 5$ never decreases, and explain the

stationary point.

$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 \geq 0$ for all x. So the function never decreases. The only stationary point is at $x = 1$, $f(1) = 6$, where the gradient is zero but does not change sign: it is a point of inflection rather than a turning point. The graph climbs, flattens momentarily at $(1, 6)$ and then continues to climb.

Worked example 15: Optimisation with a restriction

Two numbers add to 20. Find the minimum value of the sum of their squares.

Let the numbers be x and $20 - x$. The sum of squares is $S = x^2 + (20-x)^2 = 2x^2 - 40x + 400$. $\dfrac{dS}{dx} = 4x - 40 = 0$ gives $x = 10$, $\dfrac{d^2S}{dx^2} = 4 > 0$, a minimum, and $S = 100 + 100 = 200$. The same answer comes from completing the square: $S = 2(x-10)^2 + 200$. Both methods agree, which is a good check.

Worked example 16: A gradient condition

Find the point on the curve $y = x^2 - 6x + 10$ where the tangent is parallel to the line $y = 4x - 3$.

The gradient of the curve is $\dfrac{dy}{dx} = 2x - 6$. Parallel to $y = 4x - 3$ means gradient 4, so $2x - 6 = 4$ and $x = 5$. Then $y = 25 - 30 + 10 = 5$. The point is $(5, 5)$. The tangent there is $y - 5 = 4(x - 5)$, i.e. $y = 4x - 15$, which is indeed parallel to the given line and does not coincide with it.

Method: solving an optimisation problem

1. Draw a diagram and define the variables.
2. Write down the quantity to be optimised and the constraint.
3. Use the constraint to write the quantity in terms of one variable.
4. Differentiate, set equal to zero and solve; reject values outside the allowed range.
5. Prove the nature using the second derivative (or the sign of the gradient) and answer the question in context with units.

Watch out: common mistakes in differentiation

- Forgetting to rewrite $\dfrac{1}{x^2}$ and $\sqrt{x}$ as powers before differentiating.
- Differentiating a constant term as the constant itself; it becomes 0.
- Using the gradient of the tangent as the gradient of the normal; the normal gradient is $-\dfrac{1}{m}$.

- Claiming a stationary point is a maximum or minimum without justification; if $f''=0$ you must check the sign of f' either side.
- Giving the stationary value of x when the question asks for the maximum value of the quantity.

Note: what the derivative tells you

The derivative is a gradient, a rate of change and, graphically, the slope of the tangent. Its sign tells you whether the function is increasing or decreasing; its zeros give the stationary points; and its own derivative, the second derivative, tells you how the gradient is changing and therefore the concavity. In applied problems, the derivative of displacement is velocity, the derivative of velocity is acceleration, and the derivative of a cost or profit with respect to the quantity is the marginal cost or profit. Always say what the derivative represents in the context of the question and give its units, such as metres per second or pounds per item.

Did you know?

Isaac Newton and Gottfried Leibniz independently developed calculus in the late 17th century, and their followers argued bitterly for decades over who came first. Newton called derivatives "fluxions" and used a dot above the letter, while Leibniz wrote $\frac{dy}{dx}$, the notation we still use because it shows the idea of a tiny change in y divided by a tiny change in x. Engineers use calculus to design everything from bridges to the shape of a drinks can.

Otto's tip

"Bring the power down, then take one off the power." And always check: is the question asking for a point (both coordinates) or only x?