Worksheet - Differentiation

Year: Year 12 · 20

Show your working.

Question 1

Question 1 – Differentiating powers of x *(Section A · Skills practice)*

Differentiate each function with respect to x.

a. $y = 5x^3 - 2x^2 + 7x - 4$ [2 marks]

b. $y = 3\sqrt{x} - \dfrac{2}{x^2}$ [3 marks]

c. $y = (x + 2)(x - 3)$ [3 marks]

d. $y = \dfrac{2x^2 - 3}{x}$ [3 marks]

Question 2

Question 2 – First principles *(Section A · Skills practice)*

Differentiate from first principles.

a. $f(x) = 3x^2$ [4 marks]

b. $f(x) = x^3$ [4 marks]

Question 3

Question 3 – Tangent and normal *(Section A · Skills practice)*

Find the equations of the tangent and the normal to each curve at the given point.

a. $y = x^2 - 3x + 1$ at $x = 2$ [5 marks]

b. $y = 2x^3 - x$ at $x = 1$ [5 marks]

Question 4

Question 4 – Increasing and decreasing functions *(Section A · Skills practice)*

$f(x) = x^3 - 6x^2 + 9x + 2$.

a. Find $f'(x)$ and factorise it. [3 marks]

b. Find the values of x for which f is increasing. [3 marks]

c. Find the values of x for which f is decreasing. [1 mark]

Question 5

Question 5 – Stationary points *(Section A · Skills practice)*

$y = x^3 - 3x^2 - 9x + 5$.

a. Find $\frac{dy}{dx}$ and the x-coordinates of the stationary points. [3 marks]
b. Find the y-coordinates. [2 marks]
c. Use $\frac{d^2y}{dx^2}$ to determine the nature of each stationary point. [3 marks]

Question 6

Question 6 – The second derivative of a quartic *(Section A · Skills practice)*

$f(x) = x^4 - 8x^2$.

a. Find $f'(x)$ and $f''(x)$. [3 marks]
b. Find the stationary points and their nature. [5 marks]
c. Find the range of values of x for which f is increasing. [3 marks]

Question 7

Question 7 – A thrown ball *(Section B · Real-life practice)*

A ball is thrown upwards from a height of 1.2 m. Its height h metres after t seconds is $h = 1.2 + 14t - 4.9t^2$.

a. Find $\frac{dh}{dt}$ and explain what it represents. [3 marks]
b. Find the time and the height at the highest point. [4 marks]
c. Find the time at which the ball lands, to 2 decimal places. [3 marks]
d. Find the velocity on landing. [2 marks]

Question 8

Question 8 – The cheapest open box *(Section B · Real-life practice)*

A company makes an open-topped box with a square base of side x metres and height h metres. The volume must be 32 m^3.

a. Show that the surface area is $S = x^2 + \frac{128}{x}$. [3 marks]
b. Find the value of x that minimises S. [4 marks]
c. Show that this is a minimum and find the minimum surface area. [3 marks]
d. The material costs £15 per square metre. Find the minimum cost. [1 mark]

Question 9

Question 9 – Maximising profit *(Section B · Real-life practice)*

A small firm makes x hundred items a week. Its weekly profit, in hundreds of pounds, is modelled by $P = -2x^2 + 60x - 300$.

a. Find $\frac{dP}{dx}$. [1 mark]
b. Find the production level giving the greatest profit and the profit at that level in pounds. [4 marks]
c. Explain why $x = 15$ is a maximum. [2 marks]
d. For which values of x is the firm making a profit? Give answers to 2 decimal places. [4 marks]

Question 10

Question 10 – A particle on a line *(Section B · Real-life practice)*

A particle moves on a straight line so that its displacement from the origin after t seconds is $s = t^3 - 9t^2 + 24t$ metres.

a. Find the velocity $v = \frac{ds}{dt}$ and the acceleration $a = \frac{dv}{dt}$. [3 marks]
b. Find the times when the particle is instantaneously at rest. [3 marks]
c. Find the displacement at each of these times. [2 marks]
d. Find the acceleration at each of these times. [2 marks]

Question 11

Question 11 – Steepest section of a rollercoaster *(Section B · Real-life practice)*

Part of a rollercoaster track is modelled by $y = x^3 - 6x^2 + 9x$, $0 \le x \le 4$, with x and y in tens of metres.

a. Find $\frac{dy}{dx}$ and the gradient at $x = 0.5$. [3 marks]
b. Find the stationary points. [4 marks]
c. Show that the gradient is a minimum when $x = 2$ and find its value. [4 marks]
d. Explain what this means for the ride. [1 mark]

Question 12

Question 12 – The cheapest can *(Section B · Real-life practice)*

A cylindrical drinks can holds 330 ml (330 cm^3). The radius is r cm and the height is h cm.

a. Show that the total surface area is $S = 2\pi r^2 + \dfrac{660}{r}$. [3 marks]
b. Find the radius that minimises S, to 2 decimal places. [4 marks]
c. Find the height and the minimum surface area. [3 marks]
d. What do you notice about h and r? [1 mark]

Question 13

Question 13 – A normal to a curve *(Section B · Real-life practice)*

A footpath follows the curve $y = x^2 - 4x + 5$ (units: 10 m). A lamp is to be placed on the normal to the path at $P(3, 2)$.

a. Show that P lies on the curve and find the gradient there. [2 marks]
b. Find the equation of the normal at P. [3 marks]
c. The normal meets the curve again at Q. Find the coordinates of Q. [4 marks]
d. Find the distance PQ. [2 marks]

Question 14

Question 14 – Population of a town *(Section B · Real-life practice)*

The population of a town (in hundreds) is modelled by $P = 2000 + 150t - 3t^2$ where t is the number of years since 2000.

a. Find the rate of change of the population at $t = 10$. [2 marks]
b. When does the population stop growing? [2 marks]
c. What is the maximum population? [2 marks]
d. Find when the model predicts the population returns to its 2000 value. [2 marks]

Question 15

Question 15 – Making an open tray *(Section C · Problem solving)*

A rectangular sheet of card measures 24 cm by 15 cm. A square of side x cm is cut from each corner and the sides are folded up to make an open tray of volume V cm^3.

x (cm)	1	2	3	4	5
V (cm^3)	286	440	486	448	350

a. Show that $V = 4x^3 - 78x^2 + 360x$. [3 marks]
b. Check the table for $x = 3$. [1 mark]
c. Find $\dfrac{dV}{dx}$ and the value of x that gives the maximum volume. [5 marks]
d. Use the second derivative to prove it is a maximum and state the maximum volume. [3 marks]

Question 16

Question 16 – Tangent that meets the curve again *(Section C · Problem solving)*

The curve C has equation $y = x^3 - 3x + 2$.

a. Find $\dfrac{dy}{dx}$ and the equation of the tangent at the point A where $x = 2$. [5 marks]
b. Show that the tangent meets C again where $x^3 - 12x + 16 = 0$. [2 marks]
c. Factorise $x^3 - 12x + 16$ and find the other point B where the tangent meets C. [5 marks]

Question 17

Question 17 – Finding a cubic from its stationary points *(Section C · Problem solving)*

The curve $y = x^3 + ax^2 + bx + c$ has stationary points at $x = -2$ and $x = 1$ and passes through the point $(1, -2.5)$.

a. Show that $\dfrac{dy}{dx} = 3(x + 2)(x - 1)$ and hence find a and b. [4 marks]
b. Find c. [2 marks]
c. Find the coordinates of the stationary points and their nature. [4 marks]
d. State the range of values of x for which y is decreasing. [1 mark]

Question 18

Question 18 – Minimum perimeter *(Section C · Problem solving)*

A rectangular allotment must have an area of 100 m^2. The width is x m.

x (m)	5	8	10	12.5	20
Perimeter (m)	50	41	40	41	50

a. Show that the perimeter is $P = 2x + \dfrac{200}{x}$. [2 marks]
b. Check two entries in the table. [2 marks]
c. Use differentiation to find the minimum perimeter and the dimensions. [5 marks]
d. The fencing costs £8.40 per metre. Find the minimum cost. [1 mark]

Question 19

Question 19 – Stationary points and inflexion *(Section C · Problem solving)*

$f(x) = x^4 - 4x^3$.

a. Show that $f'(x) = 4x^2(x - 3)$. [2 marks]
b. Find the stationary points. [3 marks]

c. Determine the nature of each using the sign of $f'(x)$ either side. [4 marks]
d. Find $f''(x)$ and the values of x where $f''(x) = 0$. [3 marks]

Question 20

Question 20 – Challenge: nearest point on a curve *(Section D · Challenge)*

The point $P(x, x^2)$ lies on the parabola $y = x^2$. The point A is $(0, 3)$.

a. Show that $AP^2 = x^4 - 5x^2 + 9$. [3 marks]
b. Find the values of x for which AP^2 is stationary and determine their nature. [6 marks]
c. Find the least distance from A to the parabola in exact form. [3 marks]