Answers - Differentiation

Year: Year 12

Answer to Question 1

a. Differentiate term by term using $\frac{d}{dx}(x^n) = nx^{n-1}$; the constant -4 differentiates to 0. $\frac{dy}{dx} = 15x^2 - 4x + 7$.

b. First rewrite the roots and fractions as powers of x. $y = 3x^{1/2} - 2x^{-2}$, so $\frac{dy}{dx} = \frac{3}{2}x^{-1/2} + 4x^{-3} = \frac{3}{2\sqrt{x}} + \frac{4}{x^3}$. Take care with the sign: $-2 \times (-2) = +4$.

c. A product of brackets cannot be differentiated term by term, so expand it first. Expand first: $y = x^2 - x - 6$, so $\frac{dy}{dx} = 2x - 1$. So the gradient function is $2x - 1$.

d. Divide term by term: $y = 2x - 3x^{-1}$, so $\frac{dy}{dx} = 2 + 3x^{-2} = 2 + \frac{3}{x^2}$. So the derivative is $2 + \frac{3}{x^2}$, which is always positive.

Answer to Question 2

a. Use the definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ and expand $(x+h)^2 = x^2 + 2xh + h^2$.
$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 - 3x^2}{h} = \frac{6xh + 3h^2}{h} = 6x + 3h$. As $h \to 0$, $f'(x) = 6x$. The term $3h$ tends to 0, which leaves the derivative $6x$, in agreement with the rule nx^{n-1}.

b. Expand $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$; the x^3 terms cancel and every remaining term has a factor h.
$\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2$. As $h \to 0$, $f'(x) = 3x^2$. So $f'(x) = 3x^2$.

Answer to Question 3

a. Find the point, then the gradient of the tangent from $\frac{dy}{dx}$; the normal is perpendicular, so its gradient is $-\frac{1}{m}$. $y(2) = -1$. $\frac{dy}{dx} = 2x - 3 = 1$ at $x = 2$.

Tangent: $y+1=x-2$, i.e. $y=x-3$. Normal gradient -1: $y+1=-(x-2)$, i.e. $y=-x+1$. The tangent is $y=x-3$ and the normal is $y=-x+1$.

b. Use the same method: point, gradient, then $y-y_1=m(x-x_1)$. $y(1)=1$. $\frac{dy}{dx}=6x^2-1=5$. Tangent: $y-1=5(x-1)$, i.e. $y=5x-4$. Normal gradient $-\frac{1}{5}$: $y-1=-\frac{1}{5}(x-1)$, i.e. $x+5y=6$. The tangent is $y=5x-4$ and the normal is $x+5y=6$.

Answer to Question 4

a. Differentiate, then take out the common factor 3 and factorise the quadratic. $f'(x)=3x^2-12x+9=3(x^2-4x+3)=3(x-1)(x-3)$. So $f'(x)=3(x-1)(x-3)$.

b. f is increasing where $f'(x)>0$; the roots of f' are 1 and 3. $f'(x)>0$ when the U-shaped quadratic is above the axis: x or $x>3$. So f is increasing for x and for $x>3$.

c. Between the roots the quadratic $3(x-1)(x-3)$ is negative. Decreasing for 1. So f is decreasing for 1.

Answer to Question 5

a. Stationary points occur where the gradient is zero. $\frac{dy}{dx}=3x^2-6x-9=3(x-3)(x+1)=0$, so $x=3$ or $x=-1$. The stationary points have x-coordinates 3 and -1.

b. Substitute each x-value into the original equation for y, not into the derivative. $x=3$: $27-27-27+5=-22$. $x=-1$: $-1-3+9+5=10$. Points $(3,-22)$ and $(-1,10)$.

c. If $\frac{d^2y}{dx^2}>0$ the point is a minimum; if it is negative, a maximum. $\frac{d^2y}{dx^2}=6x-6$. At $x=3$: $12>0$, a **minimum**. At $x=-1$: -12, a **maximum**. So $(-1,10)$ is a maximum and $(3,-22)$ is a minimum.

Answer to Question 6

a. Differentiate twice using the power rule. $f'(x)=4x^3-16x$ and $f''(x)=12x^2-16$.

b. Solve $f'(x)=0$ by taking out the factor $4x$, then use f'' to classify each point. $4x(x^2-4)=0$ gives $x=0,\ \pm 2$. $f(0)=0$; $f(\pm 2)=16-32=-16$. $f''(0)=-16$: **maximum** at $(0,0)$. $f''(\pm 2)=32>0$: **minima** at $(2,-16)$ and $(-2,-16)$. The graph is a W shape, symmetrical about the y-axis.

c. Use a sign diagram for $4x(x-2)(x+2)$ with critical values -2, 0 and 2. $f'(x)=4x(x-2)(x+2)>0$ for -2 and $x>2$. So f is increasing for -2 and for

$x > 2$.

Answer to Question 7

a. Differentiate the height with respect to time. $\frac{dh}{dt} = 14 - 9.8t$: the vertical velocity in m/s (positive upward). Its value at any time gives how fast the ball is rising.

b. At the highest point the ball is momentarily at rest, so $\frac{dh}{dt} = 0$. Highest point when $14 - 9.8t = 0$, $t = \frac{14}{9.8} =$ **1.43 s**. $h = 1.2 + 14 \times 1.4286 - 4.9 \times 1.4286^2 =$ **11.2 m**. The ball reaches its maximum height of about 11.2 m after 1.43 s.

c. The ball lands when $h = 0$; solve the quadratic and take the positive root. $4.9t^2 - 14t - 1.2 = 0$: $t = \frac{14 + \sqrt{196 + 23.52}}{9.8} = \frac{14 + 14.816}{9.8} =$ **2.94 s**. The ball lands after 2.94 s.

d. Substitute the landing time into $\frac{dh}{dt}$. $14 - 9.8 \times 2.9404 = -14.8$: **14.8 m/s downwards**. The negative sign shows that the ball is moving downwards.

Answer to Question 8

a. Use the volume to express h in terms of x; the open box has a base and four sides but no lid. $x^2 h = 32$ so $h = \frac{32}{x^2}$. Surface $= x^2 + 4xh = x^2 + \frac{128}{x}$. This is the required expression.

b. At a minimum the derivative is zero. $\frac{dS}{dx} = 2x - \frac{128}{x^2} = 0$, so $x^3 = 64$ and $x =$ **4 m** (then $h = 2$ m). So the base should be 4 m by 4 m.

c. A positive second derivative confirms a minimum. $\frac{d^2S}{dx^2} = 2 + \frac{256}{x^3} = 6 > 0$, a minimum. $S = 16 + 32 =$ **48 m**^{2}. The minimum surface area is 48 m^2.

d. Multiply the minimum area by the cost per square metre. $48 \times 15 =$ **£720**. The minimum cost is £720.

Answer to Question 9

a. Differentiate term by term. $\frac{dP}{dx} = -4x + 60$. This is the rate at which profit changes with output.

b. Set $\frac{dP}{dx} = 0$ and remember the units: x is in hundreds of items and P is in hundreds of pounds. $-4x + 60 = 0$ gives $x = 15$ (1 500 items).

$P = -2(225) + 900 - 300 = 150$, i.e. **£15 000**. The greatest profit is £15 000, made with 1 500 items a week.

c. Use the second derivative test. $\frac{d^2P}{dx^2} = -4$, so a maximum. The parabola opens downwards, so its only stationary point is the maximum.

d. Solve $-2x^2 + 60x - 300 > 0$; dividing by -2 reverses the inequality. $P > 0$: $x^2 - 30x + 150$, roots $x = 15 \pm \sqrt{75} = 15 \pm 8.66$. So 6.34. The firm makes a profit when it produces between about 634 and 2 366 items a week.

Answer to Question 10

a. Differentiate displacement to get velocity, and velocity to get acceleration. $v = 3t^2 - 18t + 24$ and $a = 6t - 18$.

b. The particle is at rest when $v = 0$. $3(t^2 - 6t + 8) = 3(t - 2)(t - 4) = 0$, so $t =$ **2 s** and **4 s**. So it is instantaneously at rest at 2 s and at 4 s.

c. Substitute these times into s. $s(2) = 8 - 36 + 48 =$ **20 m**. $s(4) = 64 - 144 + 96 =$ **16 m**.

d. Substitute the times into $a = 6t - 18$. $a(2) = -6$ m/s^2 and $a(4) = +6$ m/s^2. At $t = 2$ the particle is at its furthest from the origin (a local max of s) and at $t = 4$ it is at a local minimum before moving away again.

Answer to Question 11

a. Differentiate and substitute $x = 0.5$. $\frac{dy}{dx} = 3x^2 - 12x + 9$; at $x = 0.5$: $0.75 - 6 + 9 = 3.75$. A positive gradient means the track is rising at this point.

b. Solve $\frac{dy}{dx} = 0$ and find the y-values from the original equation. $3(x - 1)(x - 3) = 0$: $x = 1$ ($y = 4$) and $x = 3$ ($y = 0$). So $(1, 4)$ is a peak and $(3, 0)$ is the bottom of the dip.

c. To find where the gradient is smallest, differentiate the gradient function itself. The gradient function $g = 3x^2 - 12x + 9$ has $g' = 6x - 12 = 0$ at $x = 2$; $g'' = 6 > 0$, so g is smallest there: $g(2) = 12 - 24 + 9 = -3$.

d. The steepest downhill section is at $x = 2$ where the gradient is -3 (i.e. a drop of 3 units for every 1 unit across). This is the steepest descent of the ride.

Answer to Question 12

a. Use the volume to eliminate h; the can has two circular ends and a curved surface $2\pi rh$. $\pi r^2 h = 330$, so $h = \frac{330}{\pi r^2}$. $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{660}{r}$. This is the required expression.

b. Differentiate with respect to r and set the derivative equal to zero.

$\frac{dS}{dr} = 4\pi r - \frac{660}{r^2} = 0 \Rightarrow r^3 = \frac{660}{4\pi} = 52.52 \Rightarrow r =$ **3.74 cm**. ($\frac{d^2S}{dr^2} = 4\pi + \frac{1320}{r^3} > 0$ so a minimum.) The surface area is smallest when the radius is 3.74 cm.

c. Substitute the unrounded value $r = 3.745$ to avoid rounding errors. $h = \frac{330}{\pi \times 3.745^2} =$ **7.49 cm**. $S = 2\pi(14.02) + \frac{660}{3.745} = 88.1 + 176.2 =$ **264.4 cm**^{2}. The height is 7.49 cm and the minimum surface area is about 264 cm^2.

d. $h \approx 2r$: the best can has height equal to its diameter. Real cans are taller than this because of manufacturing and design considerations.

Answer to Question 13

a. Substitute $x = 3$ into the curve, then differentiate. $9 - 12 + 5 = 2$ ✓. $\frac{dy}{dx} = 2x - 4 = 2$. So P is on the curve and the gradient there is 2.

b. The normal is perpendicular to the tangent, so its gradient is $-\frac{1}{2}$. Normal gradient $-\frac{1}{2}$: $y - 2 = -\frac{1}{2}(x - 3)$, so $x + 2y = 7$. The normal at P is $x + 2y = 7$.

c. Solve the equations of the curve and the normal simultaneously; $x = 3$ must be one root because P lies on both. $x^2 - 4x + 5 = \frac{7 - x}{2} \Rightarrow 2x^2 - 7x + 3 = 0 \Rightarrow (2x - 1)(x - 3) = 0$. The other root is $x = \frac{1}{2}$, $y = \frac{7 - 0.5}{2} = 3.25$. So $Q = (0.5,\ 3.25)$.

d. Use Pythagoras with $\Delta x = 2.5$ and $\Delta y = 1.25$. $PQ = \sqrt{2.5^2 + 1.25^2} = \sqrt{7.8125} = 2.80$ units = **28 m**. So the lamp at Q is about 28 m from P.

Answer to Question 14

a. Differentiate P with respect to t and substitute $t = 10$. $\frac{dP}{dt} = 150 - 6t$; at $t = 10$: **90 hundred per year** (9 000 people per year). In 2010 the population is increasing at about 9 000 people per year.

b. The population stops growing when $\frac{dP}{dt} = 0$. $150 - 6t = 0$ gives $t = 25$, i.e. in **2025**. The population stops growing in 2025.

c. Substitute $t = 25$ into P; $\frac{d^2P}{dt^2} = -6$ confirms a maximum. $P(25) = 2000 + 3750 - 1875 = 3875$ hundred, i.e. **387 500** people. The maximum population is 387 500.

d. Solve $P = 2000$, the value at $t = 0$, and reject the root $t = 0$. $150t - 3t^2 = 0$ gives

$t = 50$: **2050**. The model predicts the population is back to its 2000 value in 2050.

Answer to Question 15

a. After cutting out the squares, the base measures $(24 - 2x)$ by $(15 - 2x)$ and the height is x. $V = x(24 - 2x)(15 - 2x) = x(360 - 78x + 4x^2) = 4x^3 - 78x^2 + 360x$. This is the required formula.

b. Substitute $x = 3$: the base is 18 cm by 9 cm. $V(3) = 3 \times 18 \times 9 = 486$ ✓. The table value is correct.

c. Differentiate, set $\frac{dV}{dx} = 0$ and factorise; x must be less than half of 15 cm. $\frac{dV}{dx} = 12x^2 - 156x + 360 = 12(x^2 - 13x + 30) = 12(x - 3)(x - 10)$. $x = 3$ or 10; but x so $x =$ **3**. So squares of side 3 cm give the largest tray.

d. Substitute $x = 3$ into the second derivative. $\frac{d^2V}{dx^2} = 24x - 156 = -84$ at $x = 3$: a maximum. $V_{max} =$ **486 cm**^{3}. The largest possible tray holds 486 cm^3.

Answer to Question 16

a. Find the gradient and the point at $x = 2$, then use $y - y_1 = m(x - x_1)$. $\frac{dy}{dx} = 3x^2 - 3 = 9$ at $x = 2$; $y = 8 - 6 + 2 = 4$. Tangent: $y - 4 = 9(x - 2)$, i.e. $y = 9x - 14$. The tangent at $A(2, 4)$ is $y = 9x - 14$.

b. The tangent meets the curve where the two expressions for y are equal. $x^3 - 3x + 2 = 9x - 14 \Rightarrow x^3 - 12x + 16 = 0$. This is the required cubic equation.

c. Since the line is tangent at $x = 2$, $(x - 2)^2$ is a factor: $x^3 - 12x + 16 = (x - 2)^2(x + 4)$. The other root is $x = -4$, $y = 9(-4) - 14 = -50$. So $B = (-4, -50)$. (Check on C: $-64 + 12 + 2 = -50$ ✓.) So the tangent meets the curve again at $B(-4, -50)$.

Answer to Question 17

a. Stationary points where $\frac{dy}{dx} = 3x^2 + 2ax + b = 0$ at $x = -2$ and $x = 1$, so $\frac{dy}{dx} = 3(x + 2)(x - 1) = 3x^2 + 3x - 6$. Compare: $2a = 3$, $a =$ **1.5**; $b = -6$. So $a = 1.5$ and $b = -6$.

b. Substitute the point into the curve. At $(1, -2.5)$: $1 + 1.5 - 6 + c = -2.5$, so $c =$ **1**.

c. Substitute the x-values into y and use the second derivative test. $y = x^3 + 1.5x^2 - 6x + 1$. $x = -2$: $-8 + 6 + 12 + 1 = 11$; $\frac{d^2y}{dx^2} = 6x + 3 = -9$:

maximum $(-2, 11)$. $x = 1$: -2.5; second derivative $9 > 0$: **minimum** $(1, -2.5)$.

d. For a positive cubic, y decreases between the maximum and the minimum. Decreasing for -2.

Answer to Question 18

a. If the width is x, the length is the area divided by the width. Length $= \frac{100}{x}$. $P = 2\left(x + \frac{100}{x}\right) = 2x + \frac{200}{x}$. This is the required formula.

b. Substitute two values of x into the formula. $x = 8$: $16 + 25 = 41$ ✓. $x = 20$: $40 + 10 = 50$ ✓. Both entries agree with the table.

c. Differentiate, set the derivative equal to zero and keep the positive root. $\frac{dP}{dx} = 2 - \frac{200}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = 10$. $\frac{d^2P}{dx^2} = \frac{400}{x^3} > 0$: minimum. The shape is a 10×10 square and $P =$ **40 m**. The minimum perimeter is 40 m.

d. Multiply the minimum perimeter by the cost per metre. $40 \times 8.40 =$ **£336**. The minimum cost is £336.

Answer to Question 19

a. Differentiate and take out the common factor $4x^2$. $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. This is the required result.

b. Solve $f'(x) = 0$; the factor x^2 gives a repeated root at 0. $x = 0$ ($f = 0$) and $x = 3$ ($f = 81 - 108 = -27$). Points $(0, 0)$ and $(3, -27)$.

c. Since $4x^2 \geq 0$, the sign of f' is the sign of $(x - 3)$ on both sides of $x = 0$: negative on both sides. So $(0, 0)$ is a **point of inflection** (stationary, not a turning point). At $x = 3$ the sign changes from negative to positive: **minimum**. So $(3, -27)$ is a minimum and $(0, 0)$ is a stationary point of inflection.

d. Differentiate f' again and factorise. $f''(x) = 12x^2 - 24x = 12x(x - 2) = 0$ at $x = 0$ **and** $x = 2$; the curve changes concavity there.

Answer to Question 20

a. Use Pythagoras with the horizontal distance x and the vertical distance $x^2 - 3$. $AP^2 = x^2 + (x^2 - 3)^2 = x^2 + x^4 - 6x^2 + 9 = x^4 - 5x^2 + 9$. This is the required expression.

b. Minimising AP is the same as minimising AP^2, which avoids differentiating a square root. Let $f = x^4 - 5x^2 + 9$. $f' = 4x^3 - 10x = 2x(2x^2 - 5) = 0$ gives $x = 0$ or $x = \pm\sqrt{2.5} = \pm 1.58$. $f'' = 12x^2 - 10$. At $x = 0$: -10 (maximum of AP^2, i.e. a local farthest point). At $x = \pm 1.58$: $f'' = 20 > 0$ (minima). So there are minima at $x = \pm\sqrt{2.5}$ and a local maximum at $x = 0$.

c. Substitute $x^2 = 2.5$ into AP^2 and take the square root.

$f(\pm\sqrt{2.5}) = 6.25 - 12.5 + 9 = 2.75 = \frac{11}{4}$. Least distance $= \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2} \approx 1.66$.

So the least distance from A to the parabola is $\frac{\sqrt{11}}{2}$.