Test answers - Binomial expansion and sequences

Year: Year 12 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Use the binomial expansion with $a = 2$, $b = -3x$ and $n = 4$; the coefficients are 1, 4, 6. $16 + 4 \times 8(-3x) + 6 \times 4(9x^2) = 16 - 96x + 216x^2$ (3). $16 - 0.96 + 0.0216 = 15.0616$ (1).

Since $2 - 3x = 1.97$ we need $x = 0.01$. The estimate is $1.97^4 \approx 15.0616$; the calculator gives $15.0614\ldots$, so the estimate is good. Marks: 1 for each correct term, 1 for the substitution.

Common errors: forgetting to raise 2 to the powers 3 and 2, or losing the minus sign of $-3x$, which turns the x coefficient into $+96$. The x^2 coefficient is positive because $(-3x)^2 = 9x^2$.

Answer to Question 2 (4 marks)

Expand $(1 + 2x)^5$ as far as x^3; the x^3 term of the product comes from two pairs: $80x^3 \times 1$ and $40x^2 \times (-x)$.

$(1 + 2x)^5 = 1 + 10x + 40x^2 + 80x^3 + \ldots$ (1). Multiply by $(1 - x)$: the x^3 term is $80x^3 - 40x^3 = 40x^3$ (2 for the products, 1 for the answer 40).

So the coefficient of x^3 is 40. Marks: 1 for the expansion, 2 for the two correct products, 1 for the final answer.

Answer to Question 3 (4 marks)

a. Write $u_3 = a + 2d = 11$ and $u_8 = a + 7d = 31$, then subtract to eliminate a. $5d = 20$, $d = 4$, $a = 3$ (2) Check: $3 + 2 \times 4 = 11$ ✓ and $3 + 7 \times 4 = 31$ ✓. The first term is 3 and the common difference is 4.

b. The 20th term is $3 + 19 \times 4 = 79$, and $S_{20} = \frac{20}{2}[2 \times 3 + 19 \times 4]$. $S_{20} = 10(6 + 76) = 820$ (2) So the sum of the first 20 terms is 820.

Alternative for (b): $S_n = \frac{n}{2}(a + l)$ with last term $l = 79$ gives $10 \times (3 + 79) = 820$, the same total. The method mark is awarded for a correct sum formula with $a = 3$ and $d = 4$ substituted.

Answer to Question 4 (4 marks)

a. Since $|r| = \frac{2}{3} < 1$ the series converges and $S_\infty = \dfrac{a}{1 - r}$. $\dfrac{18}{1/3} = 54$ (2) The sum

to infinity is 54.

b. Use $S_n = 54\left(1 - \left(\frac{2}{3}\right)^n\right)$ and take logarithms; dividing by $\ln\frac{2}{3} < 0$ reverses the inequality. $54\left(1 - \left(\frac{2}{3}\right)^n\right) > 53$ gives $\left(\frac{2}{3}\right)^n < \frac{1}{54}$ (1), $n > 9.84$ so $n = 10$ (1) Because n must be a whole number, the smallest value is $n = 10$. Check: $\left(\frac{2}{3}\right)^{10} \approx 0.0173 < \frac{1}{54} \approx 0.0185$, but $\left(\frac{2}{3}\right)^9 \approx 0.0260$ is too large ✓.

In (a), the first mark is for quoting the formula with $a = 18$ and $r = \frac{2}{3}$, the second for 54. In (b), trial and improvement is also accepted if both $S_9 \approx 52.60$ and $S_{10} \approx 53.06$ are shown. Since 53 is very close to the limit 54, many terms are needed before the partial sums pass it.

Answer to Question 5 (4 marks)

a. The weekly amounts form an arithmetic sequence with $a = 2$ and $d = 1$, so $u_{25} = 2 + 24 \times 1$. £26 (1) In week 25 the child saves £26.

b. Use $S_n = \frac{n}{2}(a + l)$. $\frac{25}{2}(2 + 26) = £350$ (1) After 25 weeks the total is £350.

c. Form $S_n = \frac{n}{2}[4 + (n - 1)] = \frac{n}{2}(n + 3)$ and test whole numbers near the solution of $n^2 + 3n = 1000$. $S_n = \frac{n}{2}(n + 3) > 500$, so $n^2 + 3n > 1000$ (1); $n = 30$ gives 990, $n = 31$ gives 1054, so 31 weeks (1) The total first exceeds £500 after 31 weeks.