Lesson - Binomial expansion and sequences

Year: Year 12

This chapter covers two linked ideas: expanding brackets raised to a power quickly, and working with ordered lists of numbers. Both depend on spotting patterns and writing them with algebra.

Starter activity

A pound coin is placed on the first square of a chessboard, two pounds on the second, four on the third, doubling every square. A different rule puts one pound on the first square, two on the second, three on the third and so on. Without a calculator guess which rule gives more money on the 64th square and in total. Then work out the first ten squares of each rule. How early can you see which rule pulls ahead?

Pascal's triangle and $\binom{n}{r}$

Definition: binomial coefficient

$\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$ is the number of ways of choosing r items from n. The coefficients of $(1+x)^n$ form Pascal's triangle: each number is the sum of the two above it.

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1

Key fact: values of $\binom{n}{r}$

$\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = n$ and $\binom{n}{r} = \binom{n}{n-r}$. For example $\binom{8}{3} = \dfrac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$.

The binomial expansion

Theorem: binomial expansion

For a positive integer n:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n.$$

The powers of a go down, the powers of b go up, and each term has total power n.

Method: expanding $(a+bx)^n$

1. Write bx in brackets, with its sign: $(2-3x)^4$ has $b=-3$.
2. List each term: coefficient $\binom{n}{r}$, then a^{n-r}, then $(bx)^r$.
3. Simplify each term afterwards.

Worked example 1

Expand $(2-3x)^4$.

Terms: $2^4 = 16$; $4 \times 2^3 \times (-3x) = -96x$; $6 \times 2^2 \times (9x^2) = 216x^2$; $4 \times 2 \times (-27x^3) = -216x^3$; $(-3x)^4 = 81x^4$. So $(2-3x)^4 = 16 - 96x + 216x^2 - 216x^3 + 81x^4$.

Worked example 2

Find the coefficient of x^2 in $(1+x)(1+2x)^5$.

$(1+2x)^5 = 1 + 10x + 40x^2 + \cdots$. The x^2 terms come from $1 \times 40x^2$ and $x \times 10x$: $40 + 10 = 50$.

Worked example 3

Use the expansion to approximate 1.02^5.

$1.02^5 = (1+0.02)^5 \approx 1 + 5(0.02) + 10(0.02)^2 + 10(0.02)^3 = 1.10408$.

Watch out

Brackets around a negative: $(-3x)^2 = 9x^2$, not $-9x^2$. The powers of the number inside the bracket multiply in too, not just $\binom{n}{r}$.

Sequences and sigma notation

Definition: sequence and sigma notation

A sequence is defined by a **term-to-term rule** such as $u_{n+1} = 2u_n - 1$ with $u_1 = 3$ (giving $3, 5, 9, 17, 33, \ldots$), or by a formula for the nth term. It can be *increasing*, *decreasing* or *periodic*. Sigma notation $\sum_{r=1}^{n} u_r$ means $u_1 + u_2 + \cdots + u_n$.

Arithmetic sequences

Definition: arithmetic sequence

Each term adds a **common difference** d.

Rule: arithmetic formulae

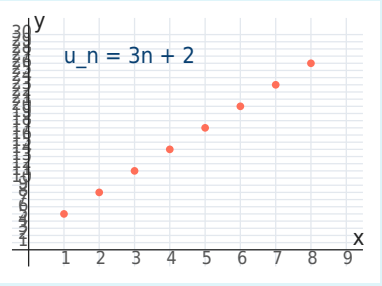
$u_n = a + (n-1)d$ and $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + l)$, where l is the last term.

Proof: pair the first and last term; each pair adds to $a + l$ and there are $\frac{n}{2}$ pairs.

Worked example 4

For $5, 8, 11, \ldots$ find u_{20} and S_{20}.

$u_n = 5 + 3(n-1) = 3n + 2$, so $u_{20} = 62$, and $S_{20} = 10(10 + 57) = 670$.



Geometric sequences

Definition: geometric sequence

Each term is multiplied by a **common ratio** r.

Rule: geometric formulae

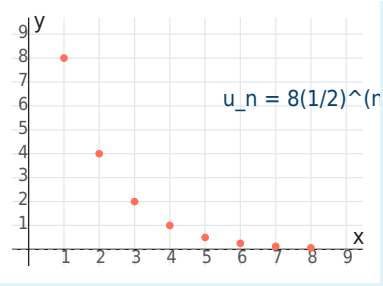
$u_n = ar^{n-1}$ and $S_n = \frac{a(1-r^n)}{1-r}$. Proof: $S_n - rS_n = a - ar^n$, so $S_n(1-r) = a(1-r^n)$.

If $|r|$ the sum converges to $S_\infty = \frac{a}{1-r}$.

Worked example 5

Find S_∞ for $8, 4, 2, 1, \ldots$.

$r = \frac{1}{2}$ so $S_\infty = \frac{8}{1 - \frac{1}{2}} = 16$.



Worked example 6

A geometric sequence has $u_2 = 12$ and $u_5 = -96$. Find a and r.

$r^3 = \dfrac{-96}{12} = -8$, so $r = -2$, and $a = \dfrac{12}{-2} = -6$.

Worked example 7

For $4 + 7 + 10 + \cdots$ find the smallest n with $S_n > 500$.

$S_n = \dfrac{n}{2}(8 + 3(n-1)) = \dfrac{n}{2}(3n + 5)$. $S_{17} = 476$ and $S_{18} = 531$, so $n = 18$.

Worked example 8

A ball bounces back to $\frac{3}{4}$ of its previous height; the first rebound is 2 m. Find the 5th rebound and the total of all rebounds.

Geometric with $a = 2$, $r = \frac{3}{4}$. The 5th rebound is $2 \times \left(\frac{3}{4}\right)^4 \approx 0.63$ m, and the total is $\dfrac{2}{1 - \frac{3}{4}} = 8$ m.

Worked example 9

Write $3 + 5 + 7 + \cdots + 41$ in sigma notation and find its value.

The rth term is $2r + 1$ and the last term 41 gives $r = 20$, so the sum is $\sum_{r=1}^{20}(2r + 1) = \dfrac{20}{2}(3 + 41) = 440$.

Watch out

Use $n - 1$, not n, in $u_n = a + (n-1)d$. Never use S_∞ when $|r| \geq 1$. Do not confuse "the nth term" with "the sum of n terms". For contexts such as savings or rebounds, say first whether the model is arithmetic or geometric and why.

Worked example 10: Expanding with a negative or fractional coefficient

Expand $\left(2x - \frac{1}{x}\right)^4$ fully and find the coefficient of x^2.

Use $\binom{4}{r}(2x)^{4-r}\left(-\frac{1}{x}\right)^r$. $r = 0$: $16x^4$. $r = 1$: $4 \times 8x^3 \times \left(-\frac{1}{x}\right) = -32x^2$. $r = 2$: $6 \times 4x^2 \times \frac{1}{x^2} = 24$. $r = 3$: $4 \times 2x \times \left(-\frac{1}{x^3}\right) = -\frac{8}{x^2}$. $r = 4$: $\frac{1}{x^4}$. So the expansion is $16x^4 - 32x^2 + 24 - 8x^{-2} + x^{-4}$ and the coefficient of x^2 is -32. Note how the signs alternate because the second term is negative.

Worked example 11: Two unknowns from two conditions

The 2nd term of an arithmetic series is 7 and the sum of the first 10 terms is 175. Find a and d.

$a + d = 7$ and $S_{10} = 5(2a + 9d) = 175$, so $2a + 9d = 35$. From the first equation $a = 7 - d$. Substitute: $14 - 2d + 9d = 35$, so $7d = 21$ and $d = 3$, then $a = 4$. Check: the sequence is 4, 7, 10, ... and $S_{10} = \frac{10}{2}(2 \times 4 + 9 \times 3) = 5 \times 35 = 175$ ✓. Whenever a question gives two pieces of information about an arithmetic or geometric sequence, write one equation for each and solve them simultaneously.

Worked example 12: Sum of a geometric series to find n

A geometric sequence has $a = 5$ and $r = 3$. Find the smallest n for which the sum of the first n terms exceeds 10 000.

$S_n = \frac{5(3^n - 1)}{2} > 10000$, so $3^n - 1 > 4000$ and $3^n > 4001$. Taking logs: $n > \frac{\ln 4001}{\ln 3} = 7.55$, so $n = 8$. Check: $S_7 = \frac{5(2187 - 1)}{2} = 5465$ and $S_8 = \frac{5(6561 - 1)}{2} = 16\,400$. So $n = 8$. When the ratio exceeds 1, the sum grows very quickly, so large sums are reached in few steps.

Worked example 13: Sigma notation with a geometric series

Evaluate $\sum_{r=1}^{\infty} 8\left(\frac{1}{4}\right)^r$ and $\sum_{r=1}^{6} 2^r$.

The first is geometric with $a = 8 \times \frac{1}{4} = 2$ and $r = \frac{1}{4}$, so $S_\infty = \frac{2}{1 - \frac{1}{4}} = \frac{8}{3}$. The second has $a = 2$, $r = 2$, $n = 6$: $S_6 = \frac{2(2^6 - 1)}{2 - 1} = 126$. Check by adding: $2 + 4 + 8 + 16 + 32 + 64 = 126$ ✓. Always find the first term by substituting the first value of r, not by assuming it is the coefficient in front.

Worked example 14: Why the binomial series is an approximation

Use the expansion of $(1+x)^{10}$ to estimate 1.001^{10} and explain why two terms are enough.

$(1+x)^{10} = 1 + 10x + 45x^2 + \ldots$ With $x = 0.001$ the terms are 1, 0.01 and 0.000045. Two terms give 1.01 and three give 1.010045. The true value is $1.01004512\ldots$, so two terms already give 3 significant figures correct and three terms give 6. A smaller x makes the later terms shrink much faster, because each term has an extra factor of x. For x close to 1, however, many terms would be needed.

Worked example 15: Proving a formula for the sum of an arithmetic series

Prove that the sum of the first n terms of an arithmetic series is $S_n = \frac{n}{2}[2a + (n-1)d]$.

Write the sum forwards and backwards: $S_n = a + (a+d) + \cdots + (a+(n-1)d)$ and $S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + a$. Add the two lines term by term. Each pair of terms adds to $2a + (n-1)d$, and there are n pairs, so $2S_n = n[2a + (n-1)d]$ and $S_n = \frac{n}{2}[2a + (n-1)d]$. This is the method that Gauss is said to have used as a schoolboy to add the numbers from 1 to 100 in seconds: 50 pairs, each adding to 101, give 5050.

Worked example 16: Using the sum to infinity to find the first term

A geometric series has common ratio $\frac{2}{5}$ and sum to infinity 50. Find the first term and the sum of the first three terms.

$S_\infty = \dfrac{a}{1-r}$, so $50 = \dfrac{a}{3/5}$ and $a = 50 \times \frac{3}{5} = 30$. The first three terms are 30, 12 and 4.8, with sum 46.8. Check with the formula: $S_3 = \dfrac{30(1-0.4^3)}{0.6} = 50 \times 0.936 = 46.8$ ✓.

The remaining terms after the third add up to $50 - 46.8 = 3.2$, which is the sum to infinity of a series starting at 1.92 with the same ratio: $\dfrac{1.92}{0.6} = 3.2$ ✓.

Note: choosing the right strategy

Most sequence problems give you two pieces of information about the sequence, such as two terms or a term and a sum. Translate each piece into an equation using the formulae, and then solve the equations simultaneously. For a geometric sequence, dividing one equation by the other eliminates a and leaves an equation in r alone. For an arithmetic sequence, substitution or subtraction removes a or d. When a question asks for the first value of n satisfying an inequality, solve the equation as if it were an equality, then test the integers either side to be sure, because rounding errors can lead to the wrong whole number.

Method: choosing between sequence formulae

1. Decide whether the sequence adds a constant (arithmetic) or multiplies by a constant (geometric).
2. For arithmetic: $u_n = a + (n-1)d$ and $S_n = \frac{n}{2}[2a + (n-1)d]$ or $\frac{n}{2}(a + l)$.
3. For geometric: $u_n = ar^{n-1}$, $S_n = \dfrac{a(1-r^n)}{1-r}$ and $S_\infty = \dfrac{a}{1-r}$ only when $|r|$.
4. To find n in a geometric problem, take logarithms; in an arithmetic one solve a quadratic and check the neighbouring integers.

Watch out: common sequence and binomial errors

- Writing the nth term as $a + nd$. It is $a + (n-1)d$.
- Using S_∞ when $|r| \geq 1$; the series diverges.
- Forgetting to raise the coefficient to the power: in $(2 + 3x)^4$ the x^2 term involves $(3x)^2 = 9x^2$.
- Mixing up the term number r and the power: the term containing x^r in $(1 + x)^n$ is $\binom{n}{r}x^r$, but the $(r+1)$th term.
- Losing a minus sign: in $(1 - 2x)^7$ every odd power has a negative coefficient.

Note: arithmetic or geometric growth in the real world

An arithmetic sequence models growth by a fixed amount, such as a weekly saving that rises by £5, while a geometric sequence models growth by a fixed percentage, such as compound interest. Over a short time the two can look similar, but a geometric sequence with ratio greater than 1 always overtakes an arithmetic one eventually, as the job offer problem shows (year 17). A geometric sequence with ratio between 0 and 1 shrinks, and its sum to infinity gives the limit of a process like the bouncing ball or the filling tank. Recognising which model fits is the first step in any sequence problem.

Did you know?

According to legend, the inventor of chess asked the king for rice: one grain on the first square, two on the second, four on the third, and so on. The 64 squares would need $2^{64} - 1$ grains, about 18 quintillion, more rice than has been harvested in human history. The binomial theorem has a long history too: Pascal\'s triangle was known to Chinese mathematicians such as Yang Hui in the 13th century, 400 years before Blaise Pascal wrote about it.

Otto's tip

To choose a formula ask: do I add (arithmetic) or multiply (geometric) to get from one term to the next?