Worksheet - Binomial expansion and sequences

Year: Year 12 · 20

Show your working.

Question 1

Question 1 – Expanding a binomial *(Section A · Skills practice)*

Use the binomial expansion.

a. Expand $(2+3x)^4$ in ascending powers of x. [4 marks]
b. Find the coefficient of x^3 in the expansion of $(1-2x)^7$. [3 marks]
c. Evaluate $\binom{10}{3}$ and $\binom{8}{5}$. [2 marks]

Question 2

Question 2 – Finding a particular term *(Section A · Skills practice)*

Find the required term in each expansion.

a. The term independent of x in $\left(x^2+\frac{1}{x}\right)^9$. [4 marks]
b. The constant term in $\left(x+\frac{2}{x}\right)^6$. [4 marks]
c. The coefficient of x^4 in $(3-x)^6$. [3 marks]

Question 3

Question 3 – Using the expansion to estimate *(Section A · Skills practice)*

Use the first three terms of a binomial expansion to estimate each value.

a. Write down the first three terms of $(1+x)^8$ and use $x=0.02$ to estimate 1.02^8. [3 marks]
b. Expand $(2-x)^6$ up to and including the term in x^3. [4 marks]
c. Use your answer to part (b) to estimate 1.98^6 and compare with the calculator value. [3 marks]

Question 4

Question 4 – Arithmetic sequences and series *(Section A · Skills practice)*

An arithmetic sequence begins 5, 9, 13, 17, \dots

a. Find the 20th term. [2 marks]
b. Find the sum of the first 20 terms. [2 marks]
c. Find the first term greater than 200. [3 marks]
d. Find the least n for which the sum of the first n terms exceeds 1000. [4 marks]

Question 5

Question 5 – Geometric sequences and series *(Section A · Skills practice)*

A geometric sequence has first term 3 and common ratio 2.

a. Find the 8th term. [2 marks]
b. Find the sum of the first 10 terms. [2 marks]
c. A different geometric series has first term 64 and common ratio $-\frac{1}{2}$. Show that it converges and find its sum to infinity. [3 marks]
d. Find the sum to infinity of a series whose first term is 12 and whose second term is 8. [3 marks]

Question 6

Question 6 – Sigma notation and recurrence relations *(Section A · Skills practice)*

Work with sequences defined in different ways.

a. Evaluate $\sum_{r=1}^{20}(3r+2)$. [3 marks]
b. A sequence has $u_{n+1} = 2u_n - 3$ and $u_1 = 5$. Find u_2, u_3 and u_4. [3 marks]
c. A sequence has $u_1 = 2$ and $u_{n+1} = \dfrac{1}{1-u_n}$. Show that it has period 3 and find $\sum_{n=1}^{100} u_n$. [5 marks]

Question 7

Question 7 – A regular savings plan *(Section B · Real-life practice)*

Priya starts a savings plan. She pays in £50 in the first month and increases her deposit by £5 every month.

a. How much does she deposit in month 24? [2 marks]
b. How much has she saved in total after 24 months? [2 marks]
c. Show that the total after n months is $\frac{n}{2}(95+5n)$. [2 marks]
d. After how many months does her total first exceed £5 000? [4 marks]

Question 8

Question 8 – Compound interest *(Section B · Real-life practice)*

£2 000 is invested at 3.5% compound interest per year. The value after n years is $V_n = 2000 \times 1.035^n$.

Years	0	1	2	3
Value (£)	2000	2070	2142.45	2217.44

a. Check the value after 2 years and 3 years. [2 marks]
b. Find the value after 10 years. [2 marks]
c. After how many whole years does the investment first exceed £4 000? [4 marks]
d. Identify the type of sequence and the common ratio. [2 marks]

Question 9

Question 9 – Seats in a stadium *(Section B · Real-life practice)*

The first row of a stadium stand has 20 seats. Each further row has 3 more seats than the row in front.

a. How many seats are in row 30? [2 marks]
b. How many seats are in the first 30 rows? [2 marks]
c. A concert needs 1 500 seats. What is the smallest number of rows needed? [5 marks]
d. Tickets are £35. How much would a full 30-row stand raise? [1 mark]

Question 10

Question 10 – Estimating rent increases *(Section B · Real-life practice)*

A landlord raises the monthly rent by 3% every year. The rent is currently £800. After n years the rent is 800×1.03^n.

a. Write down the first three terms of the expansion of $(1 + 0.03)^6$ and use it to estimate the rent after 6 years. [4 marks]
b. Find the exact rent after 6 years to the nearest penny. [1 mark]
c. Find the percentage error in the estimate. [2 marks]
d. Explain why the estimate is slightly low. [1 mark]

Question 11

Question 11 – The bouncing ball *(Section B · Real-life practice)*

A ball is dropped from a height of 2 m. After each bounce it rises to 70% of its previous height.

a. Find the height reached after the 5th bounce. [2 marks]
b. Find the total distance travelled by the ball before it stops bouncing. [5 marks]
c. After how many bounces is the rebound height first less than 1 cm? [4 marks]

Question 12

Question 12 – A population model *(Section B · Real-life practice)*

The number of fish in a lake at the start of year n is P_n. Each year 10% of the fish die and 50 new fish are added. So $P_{n+1} = 0.9P_n + 50$ with $P_0 = 400$.

a. Find P_1, P_2 and P_3. [3 marks]
b. Find the limit L that the population approaches, by solving $L = 0.9L + 50$. [2 marks]
c. Show that $P_n = 500 - 100 \times 0.9^n$ and check it for $n = 2$. [3 marks]
d. After how many years does the population first exceed 480? [3 marks]

Question 13

Question 13 – Estimating a power *(Section B · Real-life practice)*

Use the binomial expansion of $(3 - 2x)^5$.

a. Find the first three terms in ascending powers of x. [4 marks]
b. Use $x = 0.01$ to estimate 2.98^5. [2 marks]
c. Find the error in your estimate to 3 decimal places. [2 marks]
d. Explain why taking more terms would improve the estimate. [1 mark]

Question 14

Question 14 – A training schedule *(Section B · Real-life practice)*

A runner trains for a marathon. In week 1 she runs 8 km and each week she runs 1.5 km further than the week before.

a. How far does she run in week 12? [2 marks]
b. What is the total distance in the first 12 weeks? [2 marks]
c. In which week does she first run more than 30 km? [3 marks]
d. The race is 42.195 km. In which week does she first run further than the marathon distance? [3 marks]

Question 15

Question 15 – Two job offers *(Section C · Problem solving)*

Aisha is offered two jobs. Job A pays £24 000 in year 1 with a rise of £1 500 each year. Job B pays £22 000 in year 1 with a 5% rise each year.

Year	1	2	3	4	5
Job A (£)	24 000	25 500	27 000	28 500	30 000
Job B (£)	22 000	23 100	24 255	25 467.75	26 741.14

a. Check the Job B salary in year 3 and identify the type of sequence for each job. [3

marks]
b. Find the total earned from Job A in the first 10 years. [2 marks]
c. Find the total earned from Job B in the first 10 years. [3 marks]
d. In which year does the Job B salary first exceed the Job A salary? [5 marks]

Question 16

Question 16 – Finding n and a from coefficients *(Section C · Problem solving)*

In the expansion of $(1+ax)^n$ in ascending powers of x, the coefficient of x is -12 and the coefficient of x^2 is 60.

a. Show that $na = -12$ and $\frac{1}{2}n(n-1)a^2 = 60$. [2 marks]
b. Find n and a. [4 marks]
c. Find the coefficient of x^3. [2 marks]
d. Use the expansion with $x = 0.01$ to estimate 0.98^6 to 5 decimal places. [3 marks]

Question 17

Question 17 – A geometric series *(Section C · Problem solving)*

A geometric series has all positive terms. The second term is 12 and the fourth term is 3.

Term	1	2	3	4
Value	a	12	$12r$	3

a. Show that $r = \frac{1}{2}$ and find a. [3 marks]
b. Find the sum of the first 8 terms. [2 marks]
c. Find the sum to infinity. [2 marks]
d. Find the sum of the terms from the 5th term onwards. [3 marks]

Question 18

Question 18 – A sequence defined by recurrence *(Section C · Problem solving)*

A sequence is defined by $u_{n+1} = 3u_n - 4$ and $u_1 = k$.

a. Show that $u_3 = 9k - 16$. [2 marks]
b. Given that $u_3 = 11$ find k and list the first four terms. [3 marks]
c. Find the value L for which $u_{n+1} = u_n = L$ (a fixed point). [2 marks]
d. For $k = 3$ show that $u_n = 2 + 3^{n-1}$ and find $\sum_{n=1}^{10} u_n$. [5 marks]

Question 19

Question 19 – A tank filling slowly *(Section C · Problem solving)*

A water tank has capacity 24 litres. In the first minute 5 litres flow in. In each later minute the amount is 80% of the amount in the previous minute.

Minute	1	2	3	4	5
Total (litres)	5	9	12.2	14.76	16.808

a. Check the totals for minutes 2 and 4. [2 marks]
b. Show that the total after n minutes is $25(1 - 0.8^n)$. [2 marks]
c. Show that the tank would eventually fill, even though the flow keeps falling. [2 marks]
d. After how many minutes is the tank full? [4 marks]

Question 20

Question 20 – Challenge: arithmetic progression puzzle *(Section D · Challenge)*

Three numbers form an arithmetic sequence. Their sum is 21 and their product is 231.

a. Find the three numbers. [5 marks]
b. The sequence continues as 3, 7, 11, \dots Show that the sum of the first n terms is $S_n = 2n^2 + n$. [3 marks]
c. Find n if $S_n = 1275$. [3 marks]
d. Using $u_n = S_n - S_{n-1}$ verify that $u_n = 4n - 1$. [3 marks]