Answers - Binomial expansion and sequences

Year: Year 12

Answer to Question 1

a. Use $(a+b)^n = \sum \binom{n}{r} a^{n-r} b^r$ with $a = 2$, $b = 3x$ and $n = 4$, writing one term for each $r = 0, 1, 2, 3, 4$.
$(2+3x)^4 = 2^4 + \binom{4}{1}2^3(3x) + \binom{4}{2}2^2(3x)^2 + \binom{4}{3}2(3x)^3 + (3x)^4 = 16 + 96x + 216x^2 + 216x^3 + 81x^4$. Check: putting $x = 1$ gives $16 + 96 + 216 + 216 + 81 = 625 = 5^4$, so the coefficients are consistent.

b. Only the term with $r = 3$ contains x^3, and the sign of $-2x$ must be kept inside the bracket. The x^3 term is $\binom{7}{3}(-2x)^3 = 35 \times (-8)x^3 = -280x^3$. The coefficient is -280. A negative coefficient is expected because an odd power of -2 is negative.

c. Use $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ and cancel before multiplying. $\binom{10}{3} = \dfrac{10 \times 9 \times 8}{3 \times 2 \times 1} = \mathbf{120}$. $\binom{8}{5} = \binom{8}{3} = \dfrac{8 \times 7 \times 6}{6} = \mathbf{56}$. So $\binom{10}{3} = 120$ and $\binom{8}{5} = 56$; the symmetry $\binom{n}{r} = \binom{n}{n-r}$ saves work.

Answer to Question 2

a. Write the general term first, then choose r so that the total power of x is zero. General term: $\binom{9}{r}(x^2)^{9-r}\left(\frac{1}{x}\right)^r = \binom{9}{r}x^{18-3r}$. The power is 0 when $r = 6$: $\binom{9}{6} = \mathbf{84}$. So the term independent of x is 84, because $18 - 3r = 0$ only for $r = 6$.

b. Collect the powers of x: $x^{6-r} \times x^{-r} = x^{6-2r}$. General term: $\binom{6}{r}x^{6-r}\left(\frac{2}{x}\right)^r = \binom{6}{r}2^r x^{6-2r}$. The power is 0 when $r = 3$: $\binom{6}{3} \times 2^3 = 20 \times 8 = \mathbf{160}$. The constant term is 160; remember to raise the 2 to the power r as well as the x.

c. Here $r = 4$ gives $(-x)^4$ and the first term is $3^{6-4} = 3^2$. $\binom{6}{4} \times 3^2 \times (-x)^4 = 15 \times 9 = \mathbf{135}$. Since $(-1)^4 = 1$, the coefficient of x^4 is positive: it is 135.

Answer to Question 3

a. Use $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2}x^2 + \ldots$ with $n = 8$, so the third coefficient is $\frac{8\times7}{2} = 28$. $(1+x)^8 = 1 + 8x + 28x^2 + \ldots$. With $x = 0.02$: $1 + 0.16 + 0.0112 = \mathbf{1.1712}$. (Calculator: 1.17166.) The estimate 1.1712 is correct to 3 decimal places because the neglected terms are very small when $x = 0.02$.

b. Use $a = 2$, $b = -x$, $n = 6$: the coefficients $1, 6, 15, 20$ come from Pascal's triangle and the powers of 2 decrease from 2^6.
$(2 - x)^6 = 64 - 6 \times 32x + 15 \times 16x^2 - 20 \times 8x^3 + \cdots =$
$64 - 192x + 240x^2 - 160x^3$. The signs alternate because odd powers of $-x$ are negative.

c. Since $2 - x = 1.98$ we need $x = 0.02$, and we substitute this into the cubic from part (b). Put $x = 0.02$: $64 - 3.84 + 0.096 - 0.00128 = 60.25472$. The calculator gives $1.98^6 = 60.2547\ldots$ (agrees to 6 s.f.). So $1.98^6 \approx 60.2547$, which is extremely accurate because the next term, $60x^4 \approx 0.00001$, is tiny.

Answer to Question 4

a. Here $a = 5$ and $d = 4$, so use $u_n = a + (n - 1)d$. $u_n = 5 + 4(n - 1) = 4n + 1$, so $u_{20} =$ **81**. So the 20th term is 81.

b. Use $S_n = \frac{n}{2}(a + l)$ with first term 5 and last term $u_{20} = 81$. $S_{20} = \frac{20}{2}(5 + 81) =$ **860**. So the sum of the first 20 terms is 860.

c. Form an inequality with the formula for u_n and solve it. $4n + 1 > 200$ gives $n > 49.75$, so $n = 50$ and the term is $u_{50} =$ **201**. Because n must be a whole number we round up: the first term greater than 200 is 201.

d. Find a formula for S_n and test the integers either side of the boundary.
$S_n = \frac{n}{2}[10 + 4(n - 1)] = n(2n + 3)$. $S_{21} = 21 \times 45 = 945$ and
$S_{22} = 22 \times 47 = 1034$. So $n =$ **22**. Since $S_{21} = 945$ and $S_{22} = 1034 > 1000$, the least value is $n = 22$.

Answer to Question 5

a. For a geometric sequence $u_n = ar^{n-1}$ with $a = 3$ and $r = 2$. $u_8 = 3 \times 2^7 =$ **384**. So the 8th term is 384.

b. Use $S_n = \dfrac{a(r^n - 1)}{r - 1}$, which is convenient when $r > 1$.
$S_{10} = \dfrac{3(2^{10} - 1)}{2 - 1} = 3 \times 1023 =$ **3069**. So the sum of the first 10 terms is 3069.

c. A geometric series converges exactly when $|r|$, so check the modulus of the ratio first. $|r| = \frac{1}{2}$, so it converges. $S_\infty = \dfrac{64}{1 - (-\frac{1}{2})} = \dfrac{64}{1.5} = \dfrac{128}{3}$. The sum to infinity is $\frac{128}{3} \approx 42.7$; the terms alternate in sign, so the partial sums oscillate towards this value.

d. Divide the second term by the first to find the common ratio. $r = \frac{8}{12} = \frac{2}{3}$.
$S_\infty = \dfrac{12}{1 - \frac{2}{3}} =$ **36**. Since $|r| = \frac{2}{3}$ the series converges and its sum to infinity is 36.

Answer to Question 6

a. Write out the first and last terms: $r = 1$ gives 5 and $r = 20$ gives 62, with 20 terms in total. An arithmetic series with first term 5, last term 62 and 20 terms: $\frac{20}{2}(5 + 62) = \mathbf{670}$. So $\sum_{r=1}^{20}(3r + 2) = 670$.

b. Substitute each term into the rule to get the next one. $u_2 = 10 - 3 = \mathbf{7}$, $u_3 = 14 - 3 = \mathbf{11}$, $u_4 = 22 - 3 = \mathbf{19}$. So $u_2 = 7$, $u_3 = 11$ and $u_4 = 19$; notice that each term is 3 more than a power of 2.

c. $u_2 = \dfrac{1}{1-2} = -1$, $u_3 = \dfrac{1}{1+1} = \frac{1}{2}$, $u_4 = \dfrac{1}{1-\frac{1}{2}} = 2 = u_1$, so the sequence repeats every 3 terms. One cycle sums to $2 - 1 + \frac{1}{2} = \frac{3}{2}$. 100 terms = 33 full cycles + 1 extra term ($u_{100} = u_1 = 2$): $33 \times \frac{3}{2} + 2 = \mathbf{51.5}$. So the sum of the first 100 terms is 51.5.

Answer to Question 7

a. The deposits form an arithmetic sequence with $a = 50$ and $d = 5$, so use $u_n = a + (n-1)d$. $u_{24} = 50 + 23 \times 5 = \mathbf{£165}$. In month 24 she deposits £165.

b. Use $S_n = \frac{n}{2}(a + l)$ with first term £50 and last term £165. $S_{24} = \frac{24}{2}(50 + 165) = 12 \times 215 = \mathbf{£2\ 580}$. After 24 months she has saved £2 580 in total.

c. Use the other form of the sum, $S_n = \frac{n}{2}[2a + (n-1)d]$. $S_n = \frac{n}{2}[2 \times 50 + (n-1) \times 5] = \frac{n}{2}(95 + 5n)$. Expanding the bracket gives $100 + 5n - 5 = 95 + 5n$, as required.

d. Set up an inequality, solve the quadratic, then check whole numbers either side. $\frac{n}{2}(95 + 5n) > 5000$ gives $5n^2 + 95n - 10000 > 0$, i.e. $n^2 + 19n - 2000 > 0$, so $n > \dfrac{-19 + \sqrt{8361}}{2} = 36.2$. Check: $S_{36} = 18 \times 275 = 4950$ and $S_{37} = 18.5 \times 280 = 5180$. So **37 months**. Her total first exceeds £5 000 after 37 months.

Answer to Question 8

a. Substitute $n = 2$ and $n = 3$ into the formula and round to the nearest penny. $2000 \times 1.035^2 = 2142.45$ ✓. $2000 \times 1.035^3 = 2217.44$ ✓. Both values in the table are therefore correct.

b. Substitute $n = 10$: $1.035^{10} = 1.41060\ldots$ $2000 \times 1.035^{10} = \mathbf{£2\ 821.20}$. After 10 years the investment is worth £2 821.20.

c. Divide both sides of $2000 \times 1.035^n > 4000$ by 2000, then take logarithms of both sides. $1.035^n > 2$, so $n > \dfrac{\ln 2}{\ln 1.035} = 20.15$. After **21 years** (value £4 119). Since

n must be a whole number we round up: the value first exceeds £4 000 after 21 years.

d. Each value is found by multiplying the previous one by the same number, 1.035. It is a geometric sequence with common ratio **1.035**. A ratio greater than 1 means the values grow.

Answer to Question 9

a. The seats per row form an arithmetic sequence with $a = 20$ and $d = 3$. $u_{30} = 20 + 29 \times 3 =$ **107**. Row 30 has 107 seats.

b. Use $S_n = \frac{n}{2}(a + l)$ with $n = 30$, $a = 20$ and $l = 107$. $S_{30} = 15(20 + 107) =$ **1 905**. The first 30 rows hold 1 905 seats.

c. Write S_n in terms of n, solve the quadratic inequality, then check the whole numbers nearby. $S_n = \frac{n}{2}[40 + 3(n-1)] = \frac{n}{2}(37 + 3n) \geq 1500$, so $3n^2 + 37n - 3000 \geq 0$, $n \geq \dfrac{-37 + \sqrt{37369}}{6} = 26.05$. $S_{26} = 13 \times 115 = 1495$ (just short) and $S_{27} = 13.5 \times 118 = 1593$. So **27 rows**. Since 26 rows give only 1 495 seats, the smallest number of rows needed is 27.

d. Multiply the number of seats by the ticket price. $1905 \times 35 =$ **£66 675**. A full 30-row stand would raise £66 675.

Answer to Question 10

a. Expand $(1 + x)^6$ with $x = 0.03$; the third coefficient is $\frac{6 \times 5}{2} = 15$. $(1 + x)^6 = 1 + 6x + 15x^2 + \ldots$ with $x = 0.03$: $1 + 0.18 + 0.0135 = 1.1935$. Rent $\approx 800 \times 1.1935 =$ **£954.80**. So the estimated rent after 6 years is about £954.80.

b. Use a calculator: $1.03^6 = 1.194052\ldots$ $800 \times 1.03^6 =$ **£955.24**. The exact rent is £955.24 to the nearest penny.

c. Percentage error $= \dfrac{\text{true} - \text{estimate}}{\text{true}} \times 100\%$. Error $= \dfrac{955.24 - 954.80}{955.24} = 0.046\%$, **about 0.05%**. The estimate is very accurate, with an error of about 0.05%.

d. Compare the signs of the terms that were left out. The remaining terms ($20x^3 + \ldots$) are all positive and have been ignored. Since $x > 0$, dropping positive terms always gives an underestimate.

Answer to Question 11

a. After each bounce the height is multiplied by 0.7, so the heights form a geometric sequence with first term 2. Heights form a geometric sequence 2×0.7^n. After 5 bounces: $2 \times 0.7^5 =$ **0.336 m**. The ball rises to 0.336 m after the 5th bounce.

b. The ball falls 2 m first; after that it goes up and comes back down each rebound height, so each rebound is counted twice. Distance $= 2 + 2 \times (1.4 + 0.98 + \ldots)$.

The series $1.4 + 0.98 + \ldots$ has $a = 1.4$, $r = 0.7$ so $S_\infty = \dfrac{1.4}{0.3} = 4.667$. Total $= 2 + 2 \times 4.667 =$ **11.33 m**. So the ball travels a total distance of about 11.33 m (exactly $\frac{34}{3}$ m).

c. 1 cm is 0.01 m, so set up an inequality in metres and solve it with logarithms. 2×0.7^n gives 0.7^n, so $n > \dfrac{\ln 0.005}{\ln 0.7} = 14.85$. After **15 bounces**. Because $\ln 0.7$ the inequality sign reverses when we divide; the rebound height is first below 1 cm after the 15th bounce.

Answer to Question 12

a. Apply the recurrence relation three times, starting from $P_0 = 400$.
$P_1 = 0.9 \times 400 + 50 = 410$, $P_2 = 0.9 \times 410 + 50 = 419$,
$P_3 = 0.9 \times 419 + 50 = 427.1$. So $P_1 = 410$, $P_2 = 419$ and $P_3 = 427.1$.

b. If the sequence converges, both P_{n+1} and P_n tend to the same limit L. $0.1L = 50$, so $L =$ **500**. The population tends to 500 fish.

c. Look for a solution of the form $P_n = L + A \times 0.9^n$ and use the initial value. Let $P_n = 500 + A \times 0.9^n$. $P_0 = 400$ gives $A = -100$. Check $n = 2$:
$500 - 100 \times 0.81 = 419$ ✓. So $P_n = 500 - 100 \times 0.9^n$, which agrees with the value found in part (a).

d. Solve the inequality using logarithms; note that dividing by $\ln 0.9$ reverses the sign. $500 - 100 \times 0.9^n > 480$ gives 0.9^n, $n > \dfrac{\ln 0.2}{\ln 0.9} = 15.3$. After **16 years**. The population first exceeds 480 after 16 years.

Answer to Question 13

a. Use $a = 3$, $b = -2x$, $n = 5$ with coefficients $1, 5, 10$.
$(3 - 2x)^5 = 3^5 + 5 \times 3^4(-2x) + 10 \times 3^3(-2x)^2 + \cdots = 243 - 810x + 1080x^2 - \ldots$
So the first three terms are $243 - 810x + 1080x^2$.

b. We need $3 - 2x = 2.98$, so $x = 0.01$; substitute into the three terms.
$3 - 2(0.01) = 2.98$, so $2.98^5 \approx 243 - 8.1 + 0.108 =$ **235.008**. So $2.98^5 \approx 235.008$.

c. Compare the estimate with the calculator value $2.98^5 = 235.00728\ldots$ Calculator: $2.98^5 = 235.007$ (3 d.p.). Error $= 235.008 - 235.007 =$ **0.001**. The error is about 0.001 to 3 decimal places (more precisely 0.00072).

d. Look at the size of the terms that were left out. Each extra term is much smaller (the next is $-720x^3 = -0.00072$) and it brings the estimate closer to the true value. So adding further terms makes the estimate converge to the exact value.

Answer to Question 14

a. The weekly distances form an arithmetic sequence with $a = 8$ and $d = 1.5$.

$u_{12} = 8 + 11 \times 1.5 =$ **24.5 km**. She runs 24.5 km in week 12.

b. Use $S_n = \frac{n}{2}(a + l)$ with $a = 8$ and $l = 24.5$. $S_{12} = 6(8 + 24.5) =$ **195 km**. She runs 195 km in total in the first 12 weeks.

c. Solve $u_n > 30$ and round up to the next whole week. $8 + 1.5(n - 1) > 30$ gives $n - 1 > 14.67$, so $n =$ **16** (30.5 km). She first runs more than 30 km in week 16.

d. Solve $u_n > 42.195$ in the same way. $8 + 1.5(n - 1) > 42.195$ gives $n - 1 > 22.8$, so $n =$ **24** (42.5 km). She first runs further than a marathon in week 24.

Answer to Question 15

a. Multiply the year 1 salary by 1.05 twice. Job B year 3: $22000 \times 1.05^2 = 24\,255$ ✓. Job A is arithmetic ($d = 1500$); Job B is geometric ($r = 1.05$). A constant difference means arithmetic; a constant ratio means geometric.

b. Use $S_n = \frac{n}{2}[2a + (n - 1)d]$ with $a = 24000$, $d = 1500$, $n = 10$. $S_{10} = 5(2 \times 24000 + 9 \times 1500) = 5 \times 61500 =$ **£307 500**. Job A pays £307 500 over 10 years.

c. Use $S_n = \dfrac{a(r^n - 1)}{r - 1}$ with $a = 22000$, $r = 1.05$, $n = 10$. $S_{10} = \dfrac{22000(1.05^{10} - 1)}{0.05} =$ **£276 714**. Job B pays about £276 714 over 10 years, so Job A pays more over this period.

d. Compare year by year with $A = 24000 + 1500(n - 1)$ and $B = 22000 \times 1.05^{n-1}$: year 16: $A = 46500$, $B = 45\,737$; year 17: $A = 48\,000$, $B = 48\,023$. So B first exceeds A in **year 17**. The geometric growth of Job B eventually overtakes the arithmetic growth of Job A, but only from year 17.

Answer to Question 16

a. Use $(1 + y)^n = 1 + ny + \dfrac{n(n-1)}{2}y^2 + \ldots$ with $y = ax$.

$(1 + ax)^n = 1 + nax + \binom{n}{2}a^2x^2 + \ldots$, so $na = -12$ and $\frac{n(n-1)}{2}a^2 = 60$. Equating coefficients with -12 and 60 gives the two equations.

b. Make a the subject of the first equation and substitute into the second. $a = -\dfrac{12}{n}$. Substitute: $\dfrac{n(n-1)}{2} \times \dfrac{144}{n^2} = 60$, so $\dfrac{72(n-1)}{n} = 60$, $72n - 72 = 60n$, $n =$ **6** and $a = -2$. Check: $na = 6 \times (-2) = -12$ ✓ and $15 \times 4 = 60$ ✓.

c. The x^3 term is $\binom{6}{3}(ax)^3$ with $a = -2$. $\binom{6}{3}(-2)^3 = 20 \times (-8) = -160$. So the coefficient of x^3 is -160.

d. We need $1 - 2x = 0.98$, so $x = 0.01$; use four terms of the expansion. $(1 - 2x)^6 = 1 - 12x + 60x^2 - 160x^3$; with $x = 0.01$: $1 - 0.12 + 0.006 - 0.00016 =$ **0.88584**. So $0.98^6 \approx 0.88584$ to 5 decimal places, which agrees with the calculator value $0.885842\ldots$

Answer to Question 17

a. Write the second and fourth terms as ar and ar^3 and divide to eliminate a. $ar = 12$ and $ar^3 = 3$. Divide: $r^2 = \frac{1}{4}$, so $r = \frac{1}{2}$ (positive terms). Then $a = 24$. So $r = \frac{1}{2}$ and the first term is 24.

b. Use $S_n = \dfrac{a(1 - r^n)}{1 - r}$ since r. $S_8 = \dfrac{24(1 - 0.5^8)}{0.5} = 48 \times 0.99609 =$ **47.8125**. The sum of the first 8 terms is 47.8125.

c. Since $|r|$ the series converges and $S_\infty = \dfrac{a}{1 - r}$. $S_\infty = \dfrac{24}{0.5} =$ **48**. The sum to infinity is 48.

d. The terms from the 5th onwards make up everything except the first four terms. $S_4 = 24 + 12 + 6 + 3 = 45$. Sum from the 5th term $= 48 - 45 =$ **3**. Alternatively, they form a geometric series with first term $\frac{3}{2}$ and ratio $\frac{1}{2}$, whose sum is also 3.

Answer to Question 18

a. Apply the recurrence twice, starting from $u_1 = k$. $u_2 = 3k - 4$, $u_3 = 3(3k - 4) - 4 = 9k - 16$. This shows that $u_3 = 9k - 16$, as required.

b. Set $9k - 16 = 11$ and solve, then generate the terms. $9k - 16 = 11$, $k = 3$. Terms: $3,\ 5,\ 11,\ 29$. So $k = 3$ and the first four terms are 3, 5, 11, 29.

c. A fixed point satisfies $L = 3L - 4$. $L = 3L - 4$, so $L =$ **2**. The fixed point is $L = 2$; the sequence moves away from it because the multiplier 3 is greater than 1.

d. Check that the formula gives $u_1 = 3$ and satisfies the recurrence, then split the sum into two parts. $u_n = 2 + 3^{n-1}$ gives $u_1 = 3$, $u_2 = 5$, $u_3 = 11$, $u_4 = 29$ and satisfies $u_{n+1} = 2 + 3^n = 3(2 + 3^{n-1}) - 4$ ✓. Sum $= 20 + \dfrac{3^{10} - 1}{2} = 20 + 29524 =$ **29 544**. The sum of the first 10 terms is 29 544.

Answer to Question 19

a. Each minute adds 80% of the previous amount: 5, 4, 3.2, 2.56 litres. Minute 2: $5 + 4 = 9$ ✓. Minute 4: $5 + 4 + 3.2 + 2.56 = 14.76$ ✓. Both totals in the table are correct.

b. The amounts form a geometric series with $a = 5$ and $r = 0.8$. Geometric: $S_n = \dfrac{5(1 - 0.8^n)}{1 - 0.8} = 25(1 - 0.8^n)$. This gives the required formula.

c. Find the sum to infinity of the series. $S_\infty = \dfrac{5}{0.2} = 25 > 24$. The total tends to 25 litres, which exceeds the capacity 24, so the tank fills. So, although the flow keeps decreasing, the total will pass 24 litres in a finite time.

d. Solve $S_n \geq 24$ with logarithms; dividing by $\ln 0.8$ reverses the inequality.

$25(1-0.8^n) \geq 24$ gives $0.8^n \leq 0.04$, $n \geq \dfrac{\ln 0.04}{\ln 0.8} = 14.4$. The tank is full during **minute 15**. The tank is full after 15 minutes.

Answer to Question 20

a. Choose symmetric unknowns so that the sum gives a at once. Let the numbers be $a-d,\ a,\ a+d$. Sum $= 3a = 21$, so $a = 7$. Product $= 7(49-d^2) = 231$, so $49-d^2 = 33$, $d = \pm 4$. The numbers are **3, 7, 11** (or 11, 7, 3). Check: $3+7+11 = 21$ and $3 \times 7 \times 11 = 231$ ✓.

b. Use $S_n = \frac{n}{2}[2a+(n-1)d]$ with $a = 3$ and $d = 4$.
$S_n = \frac{n}{2}[2 \times 3 + (n-1)4] = \frac{n}{2}(4n+2) = 2n^2+n$. This is the required formula.

c. Solve the quadratic $2n^2+n = 1275$ with the formula, keeping only the positive root. $2n^2+n-1275 = 0$ gives $n = \dfrac{-1+\sqrt{10201}}{4} = \dfrac{-1+101}{4} = \mathbf{25}$. So $n = 25$, and indeed $2 \times 625 + 25 = 1275$ ✓.

d. The nth term is the sum of n terms minus the sum of $n-1$ terms.
$u_n = (2n^2+n) - (2(n-1)^2+(n-1)) = 2n^2+n-2n^2+4n-2-n+1 = 4n-1$ ✓. This agrees with $u_n = 3+4(n-1) = 4n-1$.