Lesson - Vectors

Year: Year 11

Column vectors and vector arithmetic are on both tiers. Geometric arguments and proofs with vectors are Higher tier (marked Higher).

Starter activity

A treasure map says: "Walk 4 paces east and 3 paces north, then 2 paces west and 5 paces north." Where are you now? Draw the route on squared paper. Can you find a single walk from the start that gets you to the same place? How long is that walk? Think about how you could write each walk as a pair of numbers.

Writing vectors

Definition: vector and scalar

A **vector** has both a size (magnitude) and a direction. A **scalar** has only size. A column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ means "x across, y up". In diagrams a vector is written $\overrightarrow{AB}$ (from A to B) or as a bold letter $\mathbf{a}$; in handwriting, write a letter with a wavy line underneath, $\underset{\sim}{a}$.

Key fact: the negative of a vector

The vector $\overrightarrow{BA}$ is the negative of $\overrightarrow{AB}$: $-\overrightarrow{AB}$, the same length in the opposite direction.

Adding, subtracting and multiplying

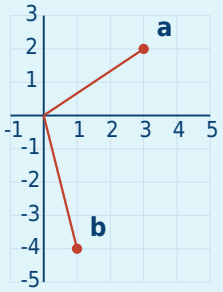
Rule: vector arithmetic

- Add or subtract the top numbers and the bottom numbers separately.
- Multiply a vector by a scalar by multiplying both numbers.

Worked example 1: arithmetic

$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. **Find** $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ **and** $2\mathbf{a} - 3\mathbf{b}$.

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $2\mathbf{a} - 3\mathbf{b} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -12 \end{pmatrix} = \begin{pmatrix} 3 \\ 16 \end{pmatrix}$.



Magnitude and vectors between points

Rule: magnitude and

The length of $\begin{pmatrix} x \\ y \end{pmatrix}$ is found with Pythagoras: $\sqrt{x^2+y^2}$. The vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ is (end minus start).

Worked example 2: magnitude

Find the magnitude of $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$, and the vector and length from $A(1,2)$ to $B(5,5)$.

The magnitude of $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ is $\sqrt{25+144} = 13$. For $A(1,2)$ and $B(5,5)$, and its length is 5.

Vector journeys and parallel vectors

Method: vector journeys

1. Travel from one point to another along any route made of known vectors, adding the vectors along the way.
2. To go backwards along a vector, use its negative.
3. Simplify by collecting like terms.

Worked example 3: triangle OAB

and . Find , the midpoint M of AB, and P on AB with $AP : PB = 1 : 2$.

. For P,

, so .

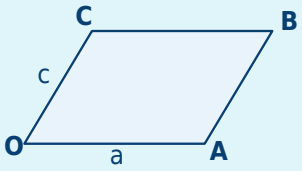
Definition: parallel vectors

Two vectors are **parallel** if one is a scalar multiple of the other. $\begin{pmatrix} 6 \\ -9 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, so they are parallel. $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$ are not, because $6 \div 2 = 3$ but $12 \div 5 \neq 3$. In algebra, $2\mathbf{a} + 4\mathbf{b}$ is parallel to $\mathbf{a} + 2\mathbf{b}$.

Worked example 4: a parallelogram

$OABC$ is a parallelogram with and . Find and .

and .



Not drawn accurately

Position vectors and ratios

Definition: position vector

A **position vector** tells you where a point is, measured from the origin O. A ratio $1:2$ on a line means $\frac{1}{3}$ of the whole line, because $1 + 2 = 3$ parts.

Worked example 5: dividing a line in a ratio

P and Q have position vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$. Find R on PQ with $PR : RQ = 1 : 2$.

. and .

Worked example 6: a resultant and simplifying

A plane flies with velocity $\begin{pmatrix}200\\0\end{pmatrix}$ km/h in still air and the wind has velocity $\begin{pmatrix}-30\\40\end{pmatrix}$ km/h. Find the resultant speed. Simplify $3(\mathbf{a}+2\mathbf{b})-2(\mathbf{a}-\mathbf{b})$.

The resultant is $\begin{pmatrix}170\\40\end{pmatrix}$, with speed $\sqrt{170^2+40^2}=\sqrt{30\,500}=174.6$ km/h. Collect like terms: $3\mathbf{a}+6\mathbf{b}-2\mathbf{a}+2\mathbf{b}=\mathbf{a}+8\mathbf{b}$.

(Higher) Proofs with vectors

Method (Higher): writing a vector proof

1. Set out the vectors you know with letters.
2. Find the vectors you need by travelling along known routes.
3. Simplify and compare.
4. Write a conclusion in words: "so MN is parallel to AB" or "so A, B and C lie on a straight line".

To prove a straight line, you need two statements: the vectors are parallel, and they share a common point.

Worked example 7 (Higher): collinear points

Show that $A(1,1)$, $B(3,2)$, $C(7,4)$ are collinear.

, . The vectors are parallel and share the point A, so A, B, C are collinear.

Worked example 8 (Higher): diagonals of a parallelogram

$ABCD$ is a parallelogram with and . Prove that the diagonals bisect each other.

and . Let M be the midpoint of AC: . Then . So M is also the midpoint of BD.

Worked example 9 (Higher): the midpoint theorem

In triangle OAB, M and N are the midpoints of OA and OB. Show that MN is parallel to AB.

[redacted]. So MN is parallel to AB and half its length.

Watch out: common mistakes

- Getting the sign wrong in [redacted] (end minus start).
- Adding magnitudes instead of the vectors.
- Writing a vector as a coordinate pair $(3, 2)$ when a column vector is needed.
- Stopping a proof without a conclusion sentence.

Worked example 10: Resultant of two journeys

Anna walks $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ km (east, north) and then $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$ km. How far is she from the start, and what is the bearing of the start from her?

Add the vectors: $\begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$. The distance is $\sqrt{4^2 + 8^2} = \sqrt{80} = 8.94$ km. For a bearing, draw a sketch: she is 4 km east and 8 km north of the start. The start is therefore south-west of her. The angle between south and the line is $\tan^{-1}\left(\frac{4}{8}\right) = 26.6°$, so the bearing of the start from Anna is $180° + 26.6° = 206.6°$.

Worked example 11: Showing that a quadrilateral is a parallelogram

$A(1, 1)$, $B(6, 2)$, $C(8, 6)$ and $D(3, 5)$. Show that $ABCD$ is a parallelogram.

Find [redacted] and [redacted]. They are equal, so AB and DC are parallel and equal in length. A quadrilateral with one pair of opposite sides equal and parallel is a parallelogram. To be thorough, also check [redacted].

Worked example 12: Scalar multiples and parallel lines

[redacted] **and** [redacted]. [redacted]. **Show that AB is parallel to OC.**

[redacted]. This equals $\frac{3}{2} \times \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, since $\frac{3}{2} \times 4 = 6$ and $\frac{3}{2} \times 2 = 3$. A

multiple of means the lines are parallel. Also $AB : OC = 3 : 2$.

Worked example 13: Midpoint using position vectors

A has position vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and B has position vector $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Find the midpoint M and the point P on AB with $AP : PB = 3 : 1$.

. For P: and $AP = \frac{3}{4}AB$, so

.

Worked example 14: Magnitude with a scalar multiple

$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Work out $|3\mathbf{a}|$ and compare it with $|\mathbf{a}|$.

$3\mathbf{a} = \begin{pmatrix} 9 \\ -12 \end{pmatrix}$, so $|3\mathbf{a}| = \sqrt{81 + 144} = \sqrt{225} = 15$. $|\mathbf{a}| = \sqrt{9 + 16} = 5$. So $|3\mathbf{a}| = 3|\mathbf{a}|$: multiplying a vector by a positive scalar k multiplies its length by k and keeps its direction. A negative scalar reverses the direction as well.

Worked example 15: A full vector proof (Higher)

$OABC$ is a parallelogram with and . Prove that the diagonals OB and AC bisect each other.

The diagonal OB has vector $\mathbf{a} + \mathbf{c}$ and the diagonal AC has vector $\mathbf{c} - \mathbf{a}$. Let X be the midpoint of OB, so . Now find the midpoint of AC using position vectors: . The two midpoints have the same position vector, so they are the same point. Therefore the diagonals meet at their common midpoint, which means they bisect each other. Notice how the proof uses only the rule for a midpoint and the fact that in a parallelogram.

Worked example 16: Reading information from a diagram

In the diagram and . Find and , and say

whether is parallel to $5\mathbf{u}+8\mathbf{v}$.

Add the two stages of the journey: . The reverse journey is the negative, so . For parallel vectors, one must be a multiple of the other. If $5\mathbf{u}+8\mathbf{v}=k(\mathbf{u}+4\mathbf{v})$ then $k=5$ from the $\mathbf{u}$ part but $k=2$ from the $\mathbf{v}$ part. These differ, so the vectors are **not parallel**.

Note: three ways to write a vector

A vector can be a column such as $\begin{pmatrix}3\\-2\end{pmatrix}$, a bold letter such as $\mathbf{a}$ (underlined $\underline{a}$ when handwritten) or a pair of points with an arrow such as . You should move freely between the three. If a diagram has a grid, read column vectors from it by counting squares across and up. If the diagram has no grid, use letters and the rules for adding and subtracting. In exams the word *magnitude* means length, and "show that" means you must write every step of the argument, not only the result.

Method: a vector proof in four steps

1. Mark every given vector on the diagram and label unknown vectors with letters such as $\mathbf{a}$ and $\mathbf{b}$.
2. Find a route from the start point to the end point that uses only known vectors. Going backwards along a vector gives its negative.
3. Simplify by collecting the $\mathbf{a}$ terms and the $\mathbf{b}$ terms.
4. To show points are collinear, show that one vector is a multiple of the other *and* that they share a common point. Write a final sentence in words.

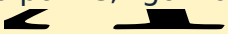
Watch out: the most common vector errors

- Writing . It should be $\mathbf{b}-\mathbf{a}$ because you go from A to B (end minus start).
- Treating a column vector as a coordinate pair when adding: a vector is a movement, not a location, unless it is a position vector.
- Showing only that two vectors are parallel when the question asks you to show that three points are in a straight line. You must also say that they share a point.
- Dividing a line in a ratio with the wrong fraction: $AP:PB=2:3$ means $AP=\frac{2}{5}AB$, not $\frac{2}{3}AB$.

Did you know?

The word *vector* comes from the Latin *vehere*, meaning "to carry". Vectors were developed in the 1880s by the American physicist Josiah Willard Gibbs and the English engineer Oliver Heaviside to describe forces and electric fields. Today the same ideas let a video game move thousands of characters across a screen and let a satellite navigation system add your speed to the speed of the wind.

Otto's tip

To find a vector between two points, "go via the origin or a point you know". Write the route as a sum, for example , and simplify.