Lesson - Transformations

Year: Year 11

At GCSE there are four transformations: reflection, rotation, translation and enlargement. Negative scale factors and combinations of transformations are Higher tier only (marked Higher).

Starter activity

Draw an arrow shaped like the letter F on squared paper. Use tracing paper to show what the shape looks like after a reflection in a vertical line, after a turn of a quarter-turn about a corner and after a slide to the right. Which of these moves changes the way the F faces, and which keeps it the same? Explain how you can tell by looking at the letter.

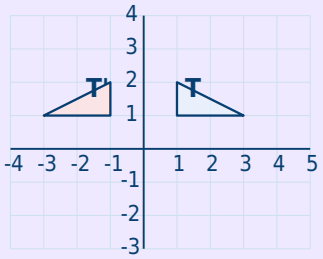
Reflection

Definition: transformation, object and image

A transformation moves or changes a shape. The original shape is the **object** and the new shape is the **image**.

Rule: reflection

To describe a reflection you must give the **mirror line**. Each point and its image are the same distance from the mirror, on opposite sides, and the line joining them is perpendicular to the mirror. Reflecting (x, y) in the x-axis gives $(x, -y)$; in the y-axis gives $(-x, y)$; in $y = x$ gives (y, x); in $y = -x$ gives $(-y, -x)$. For a mirror like $x = 4$, count the distance: the reflection of $(2, 5)$ is $(6, 5)$ because the point is 2 to the left of the line.



Rotation

Rule: rotation

To describe a rotation you must give three things: the **angle**, the **direction**

(clockwise or anticlockwise) and the **centre**. About the origin:

- 90° clockwise: $(x, y) \to (y, -x)$
- 90° anticlockwise: $(x, y) \to (-y, x)$
- 180°: $(x, y) \to (-x, -y)$

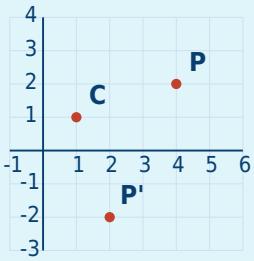
Method: rotating about another centre

1. Subtract the centre from the point.
2. Rotate using the rule for the origin.
3. Add the centre back. Tracing paper is allowed: trace the shape, put your pencil on the centre and turn the paper.

Worked example 1: rotation about another centre

Rotate $(4, 2)$ by 90° clockwise about $(1, 1)$.

Relative to the centre the point is $(3, 1)$. Rotating gives $(1, -3)$. Adding the centre back gives $(2, -2)$.



Translation

Definition: translation

A **translation** slides every point by the same amount, given as a column vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

The top number is the move right (negative means left) and the bottom number is the move up (negative means down). The shape does not turn or change size. To describe a translation, give only the vector.

Enlargement

Definition: enlargement

To describe an **enlargement** give the **scale factor** and the **centre**. Every length is multiplied by the scale factor, but angles stay the same, so the image is similar to the object. A scale factor between 0 and 1 makes the shape smaller.

Worked example 2: enlarging a point

Enlarge $A(3, 2)$ by scale factor 2 about $(1, 1)$.

The vector from the centre is $(2, 1)$, doubled to $(4, 2)$, so the image is $(5, 3)$.

Rule: scale factor and area

Lengths scale by k, but **areas** scale by k^2. A rectangle $2\text{ cm} \times 3\text{ cm}$ enlarged by scale factor 4 is 8×12, so the area is 96 cm^2, which is 6×16.

(Higher) Negative scale factors and combinations

Key fact (Higher): negative scale factors

With a negative scale factor the image is on the *opposite side* of the centre and appears upside down. The size is multiplied by the size of the scale factor, ignoring the sign. Enlarging $(4, 2)$ with scale factor -2, centre the origin, gives $(-8, -4)$. With scale factor -1 the shape is turned through 180° about the centre.

Key fact (Higher): combinations

Do the transformations in the order written, applying the first one to the original shape and the second to the new image. Some combinations equal a single transformation:

- Two reflections in parallel lines make a translation, twice as far as the distance between the mirrors and perpendicular to them.
- Two reflections in lines that meet make a rotation about the meeting point.
- Reflecting in the x-axis then the y-axis (or the other way round) is a rotation of 180° about the origin.

Describing transformations

Worked example 3: describe fully

Triangle T has vertices $(2, 1), (4, 1), (2, 2)$. Its image has vertices $(-1, 2), (-1, 4), (-2, 2)$. Describe the transformation.

Test the rule $(x, y) \to (-y, x)$: $(2, 1) \to (-1, 2)$ and $(4, 1) \to (-1, 4)$. It works. The transformation is a rotation of 90° anticlockwise about $(0, 0)$.

Worked example 4 (Higher): one transformation or two?

Point $P(2, 3)$ is reflected in the x-axis and then in the y-axis. Describe the single transformation.

The point goes to $(2, -3)$ and then to $(-2, -3)$. This is a single rotation of 180° about the origin.

Note: what you must state

Transformation	You must state
Reflection	the mirror line, such as $y = x$ or $x = 3$
Rotation	angle, direction, centre
Translation	the column vector
Enlargement	scale factor and centre

Only give *one* transformation when the question says "single transformation". Mixing words like "move and turn" gains no marks.

Watch out: common mistakes

- Leaving out the centre of a rotation or enlargement.
- Writing "move" instead of "translation".
- Confusing the vector signs: $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ is 3 left and 2 up.
- Using the scale factor for the area instead of its square.

Worked example 5: Describing a reflection

Triangle A has a vertex at $(2, 5)$ and its image has a vertex at $(8, 5)$. Describe the reflection.

The y-coordinates are equal, so the mirror line is vertical. The mirror is halfway between the two x-values: $(2 + 8) \div 2 = 5$. The reflection is in the line $x = 5$. For a reflection always give the equation of the mirror line, and check using a second point.

Worked example 6: Finding the centre of a rotation

A shape rotates 90° clockwise and the point $(5, 1)$ goes to $(3, -3)$. Find the centre of rotation.

Let the centre be (a, b). The vector from the centre to the point is $(5 - a, 1 - b)$. A clockwise quarter turn sends (x, y) to $(y, -x)$, giving $(1 - b,\ a - 5)$. Adding the centre, the image is $(a + 1 - b,\ a + b - 5) = (3, -3)$. So $a - b = 2$ and $a + b = 2$, which gives $a = 2$ and $b = 0$. The centre is $(2, 0)$. Check: the vector $(3, 1)$ turns into $(1, -3)$ and $(2, 0) + (1, -3) = (3, -3)$ as required.

Worked example 7: Using vectors for a translation

The point $(7, -2)$ is the image of $(3, 4)$ under a translation. Find the

translation vector.

The vector is image minus object: $\begin{pmatrix} 7-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$. In words, 4 units to the right and 6 units down. A translation moves every point of the shape by the same vector, so the image is congruent to the object.

Worked example 8: Enlargement with a fractional scale factor

Enlarge the triangle $(2,2)$, $(6,2)$, $(2,6)$ by scale factor $\frac{1}{2}$ with centre $(0,0)$.

Multiply every coordinate by $\frac{1}{2}$: $(1,1)$, $(3,1)$, $(1,3)$. A scale factor between 0 and 1 makes the shape smaller (a reduction) but still an enlargement in the mathematical sense. The image lies on the same side of the centre as the object.

Worked example 9: Identifying the centre from corresponding points

Under an enlargement with scale factor 3, $(2,3)$ goes to $(4,9)$. Find the centre.

Let the centre be (a,b). Then $a + 3(2-a) = 4$, so $6 - 2a = 4$ and $a = 1$. And $b + 3(3-b) = 9$, so $9 - 2b = 9$ and $b = 0$. The centre is $(1,0)$. Check: $(1,0) + 3(1,3) = (4,9)$ ✓.

Worked example 10: The effect of transformations on lengths and angles

Which transformations make a congruent image and which make a similar image?

Reflections, rotations and translations keep all lengths and angles the same, so the image is **congruent**. An enlargement keeps angles but changes lengths by the scale factor, so the image is **similar**. Reflections reverse the orientation (clockwise labels become anticlockwise), while rotations and translations keep it.

Worked example 11: A transformation described by coordinates

Describe the transformation $(x,y) \to (y,x)$ and the transformation $(x,y) \to (-y,x)$.

The first swaps the coordinates: it is a reflection in the line $y = x$. The second, $(1,0) \to (0,1)$ and $(0,1) \to (-1,0)$, is a rotation of 90° anticlockwise about the origin. Testing one or two points is the quickest way to identify a transformation.

Note: what you must state

To earn full marks when you describe a transformation, give **all** the details. For a reflection give the mirror line. For a rotation give the angle, the direction (clockwise or anticlockwise) and the centre. For a translation give the column vector. For an enlargement give the scale factor and the centre. Only one transformation should be

described when the question says "single". Use tracing paper to check the centre of a rotation.

Method: rotating about a point that is not the origin

1. Find the vector from the centre to the point: point minus centre.
2. Rotate this vector: 90° anticlockwise maps $(x, y) \to (-y, x)$, 90° clockwise maps $(x, y) \to (y, -x)$, and 180° maps $(x, y) \to (-x, -y)$.
3. Add the rotated vector to the centre to get the image.
4. Check that the image is the same distance from the centre as the original.

Watch out: errors with transformations

- A reflection needs a mirror line equation, not just the word "flip".
- Do not say "rotation" without the centre, angle and direction.
- For an enlargement, multiply the distances from the **centre**, not from the origin, unless the centre is the origin.
- Area scale factors are the square of the length scale factor, even for a negative scale factor.

Did you know?

Patterns made by reflections, rotations and translations decorate buildings all over the world. The Alhambra palace in Granada, Spain, which was built about 700 years ago, contains tilings that use all these transformations, and the Dutch artist M. C. Escher studied them there before making his famous tessellations of birds and fish. Today computer games use the same mathematics to rotate, reflect and enlarge images on a screen many times each second.

Otto's tip

Check your description by testing one vertex. If your rule sends one corner to the right place and another corner to the right place, it is almost certainly correct.