Lesson - Sine rule, cosine rule and area of a triangle

Year: Year 11

This is a Higher tier topic. SOH CAH TOA only works in right-angled triangles; for any other triangle you need the **sine rule**, the **cosine rule** and the area formula using sine. Label the triangle with capital letters for the angles and lower-case letters for the sides opposite: side a is opposite angle A.

Starter activity

You cannot walk straight across a river but you want to know how wide it is. You stand at point A on your side, look at a tree T on the opposite bank, then walk 50 m along the bank to B and look at the tree again. You can measure the angles at A and B with a protractor. Draw the triangle ABT. Which measurements do you know, and what else would you need to find the width?

The sine rule

Definition (Higher): labelling a triangle

Use capital letters for the angles and lower-case letters for the sides opposite. A **matching pair** is a side and the angle opposite it, such as a and A. The **included angle** is the angle between two known sides.

Theorem (Higher): the sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use it when you know a **matching pair** (a side and the angle opposite it) and one more piece of information.

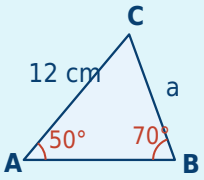
- To find a side, put the sides on top: $a = \dfrac{b \sin A}{\sin B}$.
- To find an angle, put the sines on top: $\sin A = \dfrac{a \sin B}{b}$.

If you only know two angles, find the third from $A + B + C = 180^\circ$ first.

Worked example 1 (Higher): a side

$A = 50^\circ$, $B = 70^\circ$, $b = 12$ **cm. Find** a**.**

$a = \dfrac{12 \sin 50^\circ}{\sin 70^\circ} = 9.78$ cm.



Not drawn accurately

Worked example 2 (Higher): an angle

$a = 9$, $b = 12$, $A = 40^\circ$. **Find acute angle B.**

$\sin B = \dfrac{12 \sin 40^\circ}{9} = 0.857$, so $B = \sin^{-1} 0.857 = 59.0^\circ$. Because sine is positive in two quadrants, there could be a second angle $180^\circ - 59.0^\circ = 121.0^\circ$; the question will normally say if the angle is acute or obtuse.

The cosine rule

Theorem (Higher): the cosine rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

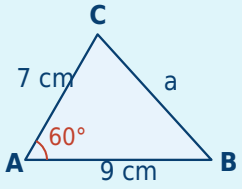
Use it when you know **two sides and the angle between them** (to find the third side), or **all three sides** (to find an angle):

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Worked example 3 (Higher): a side

$b = 7$, $c = 9$, $A = 60^\circ$. **Find a.**

$a^2 = 49 + 81 - 2 \times 7 \times 9 \times \cos 60^\circ = 130 - 63 = 67$, so $a = \sqrt{67} = 8.19$.



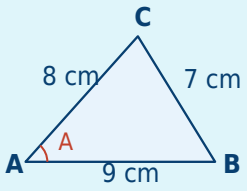
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Worked example 4 (Higher): an angle

Sides 7, 8, 9. Find the angle opposite the side of length 7. For sides 5, 6, 10

find the angle opposite 10.

$\cos A = \dfrac{8^2 + 9^2 - 7^2}{2 \times 8 \times 9} = \dfrac{96}{144} = \dfrac{2}{3}$, so $A = 48.2°$. If the angle is obtuse the cosine is negative: for sides 5, 6, 10, $\cos C = \dfrac{25 + 36 - 100}{60} = -0.65$, so $C = 130.5°$.



Not drawn accurately

Choosing the right rule

Definition (Higher): acute and obtuse angles

An **acute** angle is less than $90°$ and an **obtuse** angle is between $90°$ and $180°$. Sine is positive for both, so the sine rule can give two possible angles, but cosine is negative for an obtuse angle, so the cosine rule gives only one.

Method (Higher): choosing and using a rule

1. Sketch the triangle and label sides a, b, c opposite angles A, B, C.
2. Mark what you know and what you want to find.
3. Pick the sine rule if you have a matching pair, otherwise the cosine rule.
4. Write the formula, substitute, and check that the biggest side faces the biggest angle.

Key fact (Higher): which rule?

You know	Use
two angles and any side	sine rule
two sides and a non-included angle	sine rule
two sides and the included angle	cosine rule (find the third side)
three sides	cosine rule (find an angle)

Area of a triangle

Rule (Higher): area using sine

$$\text{Area} = \tfrac{1}{2}ab\sin C$$

Here a and b are two sides and C is the angle **between** them.

Worked example 5 (Higher): area and backwards

Find the area when $a = 8$, $b = 11$, $C = 40°$. If the area is 30 with $a = 10$, $b = 8$, find C.

Area $= \frac{1}{2} \times 8 \times 11 \times \sin 40° = 28.3$ cm^2. Backwards: $30 = 40 \sin C$, so $\sin C = 0.75$ and $C = 48.6°$ (or $131.4°$).

Worked example 6 (Higher): exact values

Find the area of an equilateral triangle of side 6 exactly. In a triangle with $AB = 4$, $AC = 6$, $A = 60°$, find BC.

Area $= \frac{1}{2} \times 6 \times 6 \times \sin 60° = 18 \times \frac{\sqrt{3}}{2} = 9\sqrt{3}$. $BC^2 = 16 + 36 - 48 \times \frac{1}{2} = 28$, so $BC = 2\sqrt{7}$.

Applying the rules to real problems

Worked example 7 (Higher): a ship and a quadrilateral

(a) A ship sails 10 km, then 14 km so that the interior angle is $110°$. Find the distance back to the start. (b) In quadrilateral $ABCD$, $AB = 6$, $BC = 8$, angle $ABC = 90°$, $CD = 7$, $AD = 5$ (cm). Find angle ADC.

(a) Cosine rule: $\sqrt{10^2 + 14^2 - 2 \times 10 \times 14 \times \cos 110°} = 19.8$ km. (b) Pythagoras in ABC: $AC = \sqrt{36 + 64} = 10$. In triangle ACD all three sides are known (5, 7, 10), so $\cos D = \frac{5^2 + 7^2 - 10^2}{2 \times 5 \times 7} = \frac{-26}{70}$, and $D = 111.8°$. The cosine is negative, so the angle must be obtuse, and it faces the longest side AC.

Key fact (Higher): why the formulas work

The cosine rule is Pythagoras' theorem with a correction term: when $A = 90°$, $\cos A = 0$ and the rule becomes $a^2 = b^2 + c^2$. If A is acute, $a^2 < b^2 + c^2$; if obtuse, $a^2 > b^2 + c^2$. The sine rule comes from the height of the triangle: height $= b \sin A = a \sin B$. The area formula comes from the same height: area $= \frac{1}{2} \times a \times b \sin C$.

Note: exam technique

- Write the formula first, then substitute: it earns a method mark even if the

arithmetic goes wrong.
- Use brackets on your calculator for $2bc\cos A$, and do not subtract before multiplying.
- When finding an angle with the cosine rule, $\cos A$ must be between -1 and 1. If not, check the numbers.
- Give angles to 1 decimal place and lengths to 3 significant figures unless told otherwise.

Watch out: common mistakes

- Using the cosine rule with an angle that is not between the two known sides.
- Calculating $2bc\cos A$ wrongly: do $2 \times b \times c \times \cos A$ in one go and subtract at the end.
- Calculator in the wrong mode.
- Forgetting to take the square root at the end of the cosine rule.
- Rounding angles too early.

Worked example 8: Choosing the right rule

In each case say which rule to use: (a) two angles and a side, (b) two sides and the included angle, (c) three sides.

(a) You know a pair of an angle and its opposite side, so use the **sine rule**. (b) You know two sides and the included angle, so use the **cosine rule** to find the third side. (c) You know all three sides, so use the **cosine rule** rearranged for an angle. A fourth case, two sides and a non-included angle, uses the sine rule and may have two solutions.

Worked example 9: Finding the area when not given the included angle

A triangle has sides 9 cm and 12 cm with the angle $A = 35^\circ$ opposite the 9 cm side. Find the area.

First use the sine rule: $\sin B = \dfrac{12\sin 35^\circ}{9} = 0.7648$, so $B = 49.9^\circ$. Then $C = 180 - 35 - 49.9 = 95.1^\circ$. The area uses the sides 9 and 12 and the angle between them, which is C: $\frac{1}{2} \times 9 \times 12 \times \sin 95.1^\circ = 53.8$ cm^2.

Worked example 10: Rearranging the sine rule

Show that the sine rule can be written $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ and use it to find an angle.

Taking the reciprocal of each fraction in $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ gives $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$. Use this form to find an angle, so that the unknown is on the top: with $a = 9$, $b = 12$ and

$A = 35^\circ$, $\sin B = \dfrac{12 \sin 35^\circ}{9}$. Use the form with the sides on top to find a side.

Worked example 11: Applying the rules in three dimensions

In a triangle formed on a hillside, $AB = 40$ m, $AC = 55$ m and angle $BAC = 38^\circ$. Find BC.

Use the cosine rule for two sides and the included angle:
$BC^2 = 40^2 + 55^2 - 2 \times 40 \times 55 \cos 38^\circ = 1600 + 3025 - 3467 = 1158$. So $BC = 34.0$ m.
Even when the triangle lies on a slope, the lengths and angles in the plane of the triangle obey the same rules.

Worked example 12: A bearing problem with two rules

A ship sails from P to Q, 14 km on bearing 065°, then from Q to R, 10 km on bearing 140°. Find PR.

The angle at Q is $180 - (140 - 65) = 105^\circ$. Cosine rule:
$PR^2 = 14^2 + 10^2 - 2 \times 14 \times 10 \cos 105^\circ = 296 + 72.5 = 368.5$, so $PR = 19.2$ km. Draw a north line at P and at Q and mark the bearing angles to find the interior angle.

Worked example 13: The ambiguous case

In triangle ABC, $a = 6$, $b = 8$ and $A = 40^\circ$. Find both possible values of B.

$\sin B = \dfrac{8 \sin 40^\circ}{6} = 0.857$, so $B = 59.0^\circ$ or $B = 180 - 59.0 = 121.0^\circ$. Both triangles exist because $40 + 121 = 161 < 180$. The ambiguous case happens when the side opposite the known angle is shorter than the other known side.

Worked example 14: Area of a triangle from three sides

A triangle has sides 13 cm, 14 cm and 15 cm. Find the area using an angle.

Find the angle opposite 14: $\cos B = \dfrac{13^2 + 15^2 - 14^2}{2 \times 13 \times 15} = \dfrac{198}{390} = 0.5077$, so $B = 59.5^\circ$.
Area $= \frac{1}{2} \times 13 \times 15 \times \sin 59.5^\circ = 84.0$ cm^2. This is the classic 13, 14, 15 triangle, whose area is exactly 84.

Note: labelling a triangle

In these formulas the side a is always opposite the angle A, side b opposite B and side c opposite C. Label the triangle before you start. The sine rule needs a matching pair (a side and its opposite angle) and the cosine rule uses an angle between two known sides. Both rules also work for obtuse triangles: the cosine of an obtuse angle is negative, so the formula for the third side then gives a longer side. Always check that the longest side is opposite the largest angle.

Method: choosing and using a rule

1. Sketch the triangle and label the known sides and angles with the correct letters.
2. Decide: a side-angle pair with one more piece of information means the sine rule; two sides with the included angle, or all three sides, means the cosine rule.
3. Write the formula, substitute, and solve. When finding an angle with the sine rule, think about whether it could be obtuse.
4. Check the result: the largest angle faces the longest side and the angles add to 180°.

Watch out: mistakes with the sine and cosine rules

- In the cosine rule, work out $2bc\cos A$ before subtracting; do not subtract $b^2 + c^2 - 2bc$ and then multiply by $\cos A$.
- Use the sine rule for the angle only when the angle is known to be acute or you have checked the obtuse case.
- For the area formula, the angle must be **between** the two sides used.
- Keep your calculator in degrees and round only at the end.

Did you know?

The sine rule and the cosine rule were used by navigators and surveyors for centuries before computers existed. In the 1800s, the Great Trigonometrical Survey of India measured a chain of triangles across the whole subcontinent with accurate angle instruments, starting from a carefully measured baseline of just a few miles. Using the sine rule repeatedly, surveyors computed the distances between mountains hundreds of kilometres apart and found the height of Mount Everest.

Otto's tip

The longest side is always opposite the largest angle. After every answer, check that your biggest side faces your biggest angle. If it does not, look for a slip.