Answers - Sine rule, cosine rule and area of a triangle

Year: Year 11

Answer to Question 1

a. $\frac{b}{\sin 65^\circ} = \frac{8}{\sin 40^\circ}$, so $b = \frac{8 \sin 65^\circ}{\sin 40^\circ} =$ **11.28 cm**. We know a side and its opposite angle (a and A) and want the side opposite B, so the sine rule links them. Multiply both sides by $\sin 65^\circ$.

b. $C = 180 - 40 - 65 = 75^\circ$. The angles of a triangle add up to 180°.

c. $c = \frac{8 \sin 75^\circ}{\sin 40^\circ} =$ **12.02 cm**. Use the sine rule again with the pair a and A, which are exact given values, rather than the rounded b.

Answer to Question 2

a. $\frac{\sin B}{12} = \frac{\sin 35^\circ}{9}$, so $\sin B = \frac{12 \sin 35^\circ}{9} = 0.7648$ and $B = 49.9^\circ$. We know a and its opposite angle A, and the side b, so the sine rule gives $\sin B$. Use $\sin^{-1}$ for the acute value.

b. $\sin B = 0.7648$ also for $180 - 49.9 = 130.1^\circ$, because $\sin x = \sin(180 - x)$. Both give valid triangles here since $35 + 130.1 < 180$. This is the ambiguous case of the sine rule: two different triangles fit the same information.

c. The acute value, 49.9°. So when the triangle is acute, angle $B = 49.9^\circ$ and angle $C = 180 - 35 - 49.9 = 95.1^\circ$ would not be acute, which shows we must check the other angles too.

Answer to Question 3

a. $a^2 = 49 + 81 - 2 \times 7 \times 9 \cos 48^\circ = 130 - 84.31 = 45.69$, so $a =$ **6.76 cm**. We know two sides and the angle between them, so we use the cosine rule.

b. $\sin B = \frac{7 \sin 48^\circ}{6.76} = 0.7696$, so $B = 50.3^\circ$. Now we know a side and its opposite angle, so we can use the sine rule. B is not the largest angle, so it is acute.

c. $C = 180 - 48 - 50.3 = 81.7^\circ$. Check with the sine rule: $\sin C = \frac{9 \sin 48^\circ}{6.76} = 0.989$, so $C = 81.7^\circ$ ✓. The angles of a triangle add up to 180°.

Answer to Question 4

a. $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$. Rearrange $c^2 = a^2 + b^2 - 2ab\cos C$ to make $\cos C$ the subject.

b. $\cos C = \dfrac{49 + 64 - 100}{2 \times 7 \times 8} = \dfrac{13}{112}$, so $C = 83.3°$. The largest angle is opposite the longest side, 10 cm. Use $\cos^{-1}$ on the calculator.

c. $\cos A = \dfrac{64 + 100 - 49}{2 \times 8 \times 10} = 0.71875$, so $A = 44.0°$. $B = 180 - 83.3 - 44.0 = 52.7°$. Use the cosine rule again for A, then subtract from $180°$ to find B.

Answer to Question 5

a. $\frac{1}{2} \times 8 \times 11 \times \sin 37° =$ **26.5 cm²**. The two sides and the included angle are known, so we can use $\frac{1}{2}ab\sin C$ directly.

b. $12 = \frac{1}{2} \times 5 \times 6 \sin C = 15 \sin C$, so $\sin C = 0.8$ and $C = 53.1°$. Substitute the area and the sides, then divide by 15 and use $\sin^{-1}$.

c. $\sin C \leq 1$, so the area is at most $\frac{1}{2} \times 5 \times 6 = 15$ cm². The largest possible value of sine is 1.

Answer to Question 6

a. $\frac{1}{2} \times 4 \times 6 \times \sin 30° = 12 \times \dfrac{1}{2} =$ **6 cm²**. Use $\sin 30° = \frac{1}{2}$ from the table of exact values.

b. $12 \sin 60° = 12 \times \dfrac{\sqrt{3}}{2} = 6\sqrt{3}$ **cm²** $= 10.39$. Use $\sin 60° = \frac{\sqrt{3}}{2}$.

c. $12 \times 1 =$ **12 cm²**. The largest area is when $\sin C = 1$, that is $C = 90°$. $\sin 90° = 1$, the largest value sine can take.

Answer to Question 7

a. $110 - 50 = 60°$. Both bearings are measured from north at P, so the angle between the two paths is the difference of the bearings.

b. $AB^2 = 12^2 + 9^2 - 2 \times 12 \times 9\cos 60° = 225 - 108 = 117$, so $AB =$ **10.82 km**. Two sides and the included angle are known, so we use the cosine rule; $\cos 60° = 0.5$.

c. $\frac{1}{2} \times 12 \times 9 \sin 60° =$ **46.8 km²**. Use $\frac{1}{2}ab\sin C$ with the two paths and the angle between them.

d. $\sin A = \dfrac{9 \sin 60°}{10.82} = 0.7205$, so $A = 46.1°$. Use the sine rule with the side AB and

its opposite angle 60°.

Answer to Question 8

a. $\frac{1}{2} \times 120 \times 150 \sin 52^\circ =$ **7 092 m²**. Two sides and the included angle are known, so we use $\frac{1}{2}ab\sin C$.

b. $c^2 = 120^2 + 150^2 - 2 \times 120 \times 150 \cos 52^\circ = 36\,900 - 22\,164 = 14\,736$, $c =$ **121.4 m**. Two sides and the included angle are known, so we use the cosine rule for the third side.

c. Perimeter $= 120 + 150 + 121.4 = 391.4$ m. Cost $= 391.4 \times 12.50 =$ **£4 892.50** (about). Add the three sides for the perimeter, then multiply by the price per metre.

Answer to Question 9

a. $180 - 62 - 71 = 47^\circ$. The angles of a triangle add up to 180°.

b. $\dfrac{AT}{\sin 71^\circ} = \dfrac{200}{\sin 47^\circ}$, so $AT =$ **258.6 m**. AT is opposite the 71° angle at B, and $AB = 200$ m is opposite the 47° angle.

c. $BT = \dfrac{200 \sin 62^\circ}{\sin 47^\circ} =$ **241.5 m**. BT is opposite the 62° angle at A.

d. Height $= 258.6 \tan 12^\circ =$ **55.0 m**. The height is opposite the 12° angle and AT is adjacent in the vertical right-angled triangle, so we use tangent.

Answer to Question 10

a. The largest angle is opposite 5.0: $\cos x = \dfrac{3.2^2 + 4.1^2 - 5^2}{2 \times 3.2 \times 4.1} = \dfrac{2.05}{26.24}$, so $x = 85.5^\circ$. All three sides are known, so we use the cosine rule in the form for $\cos x$.

b. $\frac{1}{2} \times 3.2 \times 4.1 \times \sin 85.5^\circ =$ **6.54 m²**. Use $\frac{1}{2}ab\sin C$ with the two sides next to the angle found in part (a).

c. $6 \times 6.54 \times 18 =$ **£706** (to the nearest pound). Multiply the area of one sail by 6, then by the price per square metre.

Answer to Question 11

a. At the turn the back bearing is $030 + 180 = 210^\circ$ and the new direction is 100°. The angle between them is $210 - 100 = 110^\circ$.

b. $d^2 = 250^2 + 180^2 - 2 \times 250 \times 180 \cos 110^\circ = 62\,500 + 32\,400 + 30\,782 = 125\,682$, so $d =$ **354.5 km**.

c. $\sin x = \dfrac{180 \sin 110^\circ}{354.5} = 0.4771$, so $x = 28.5^\circ$ at the start. The bearing is $030 + 28.5 = 059^\circ$ (to the nearest degree, 58.5° rounds to 059°).

Answer to Question 12

a. $AB^2 = 85^2 + 120^2 - 2 \times 85 \times 120 \cos 74^\circ = 21\,625 - 5\,623 = 16\,002$, $AB =$ **126.5 m**. Two sides and the included angle are known, so we use the cosine rule.

b. $\sin A = \dfrac{120 \sin 74^\circ}{126.5} = 0.9119$. Use the cosine rule to avoid any doubt about acute or obtuse: $\cos A = \dfrac{85^2 + 126.5^2 - 120^2}{2 \times 85 \times 126.5} = 0.4106$, so $A = 65.8^\circ$. The cosine rule gives a unique angle, because cosine is negative for obtuse angles.

c. $\frac{1}{2} \times 85 \times 120 \sin 74^\circ =$ **4 902 m²**. Use $\frac{1}{2}ab \sin C$ with the two measured sides and the angle between them.

Answer to Question 13

a. $\cos C = \dfrac{25 + 49 - 100}{2 \times 5 \times 7} = \dfrac{-26}{70}$, so $C = 111.8^\circ$. All three sides are known, so we use the rearranged cosine rule. The largest angle is opposite the longest side, 10 cm.

b. $\cos C < 0$ only for angles between 90° and 180°. So a negative cosine always means the angle is obtuse.

c. $\cos A = \dfrac{49 + 100 - 25}{2 \times 7 \times 10} = 0.8857$, $A = 27.7^\circ$. $B = 180 - 111.8 - 27.7 = 40.5^\circ$. Find A with the cosine rule, then subtract both known angles from 180°.

Answer to Question 14

a. $30 = \frac{1}{2} \times 10 \times 8 \times \sin C = 40 \sin C$, so $\sin C = 0.75$. Substitute the area and the two sides into $\frac{1}{2}ab \sin C$, then divide by 40.

b. $C = 48.6^\circ$ or $C = 180 - 48.6 = 131.4^\circ$. Sine is positive for both acute and obtuse angles, and $\sin C = \sin(180^\circ - C)$.

c. Case 1: $c^2 = 100 + 64 - 160 \cos 48.6^\circ = 164 - 105.8 = 58.2$, $c = 7.63$ cm. Case 2: $c^2 = 164 - 160 \cos 131.4^\circ = 164 + 105.8 = 269.8$, $c = 16.4$ cm. Use the cosine rule with each angle. For the obtuse angle $\cos C$ is negative, so the third side is longer.

Answer to Question 15

a. $AC^2 = 36 + 49 - 84\cos 80^\circ = 85 - 14.59 = 70.41$, so $AC =$ **8.39 cm**. In triangle ABC we know two sides and the included angle, so we use the cosine rule.

b. $\cos ACD = \dfrac{8.39^2 + 5^2 - 9^2}{2 \times 8.39 \times 5} = \dfrac{70.41 + 25 - 81}{83.9} = 0.1717$, so $ACD = 80.1^\circ$. In triangle ACD all three sides are now known, so we use the rearranged cosine rule.

c. Triangle ABC: $\frac{1}{2} \times 6 \times 7\sin 80^\circ = 20.68$. Triangle ACD: $\frac{1}{2} \times 8.39 \times 5\sin 80.1^\circ = 20.67$. Total = **41.3 cm²**. Split the quadrilateral along the diagonal AC and add the areas of the two triangles, each found with $\frac{1}{2}ab\sin C$.

Answer to Question 16

a. $b = \dfrac{6\sin 60^\circ}{\sin 45^\circ} = 6 \times \dfrac{\sqrt{3}/2}{\sqrt{2}/2} = 6 \times \dfrac{\sqrt{3}}{\sqrt{2}} = 6 \times \dfrac{\sqrt{6}}{2} = 3\sqrt{6}$. Use the exact values $\sin 60^\circ = \frac{\sqrt{3}}{2}$ and $\sin 45^\circ = \frac{\sqrt{2}}{2}$, and $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$.

b. $C = 75^\circ$. Area $= \frac{1}{2} \times 6 \times 3\sqrt{6} \times \sin 75^\circ = 9\sqrt{6}\sin 75^\circ$. The angles add up to 180°, so $C = 180 - 45 - 60$. The two sides a and b enclose angle C.

c. $3\sqrt{6} =$ **7.35 cm**. Using $\sqrt{6} \approx 2.449$, $3\sqrt{6} \approx 7.35$.

Answer to Question 17

a. $a^2 = 25 + 64 - 2 \times 5 \times 8\cos 120^\circ = 89 + 40 = 129$ (since $\cos 120^\circ = -0.5$). Two sides and the included angle are known, so we use the cosine rule. Subtracting a negative adds 40.

b. $\sqrt{129} =$ **11.36 cm**. Take the square root of 129.

c. $\frac{1}{2} \times 5 \times 8 \times \sin 120^\circ = 20 \times \dfrac{\sqrt{3}}{2} = 10\sqrt{3}$ **cm²**. Use $\frac{1}{2}ab\sin C$ with the exact value $\sin 120^\circ = \frac{\sqrt{3}}{2}$.

d. The smallest angle is opposite the 5 cm side: $\sin x = \dfrac{5\sin 120^\circ}{11.36} = 0.3812$, so $x = 22.4^\circ$. The smallest angle is opposite the shortest side. It must be acute, because the triangle already has an obtuse angle.

Answer to Question 18

a. $\cos x = \dfrac{49 + 64 - 169}{2 \times 7 \times 8} = \dfrac{-56}{112} = -\dfrac{1}{2}$, so $x = 120^\circ$. The largest angle is opposite the longest side, 13 cm. A cosine of $-\frac{1}{2}$ gives exactly 120°.

b. $\frac{1}{2} \times 7 \times 8 \sin 120^\circ = 28 \times \frac{\sqrt{3}}{2} = 14\sqrt{3}$ **cm²** $= 24.25$. Use $\frac{1}{2}ab \sin C$ with the sides 7 and 8 that enclose the 120° angle, and $\sin 120^\circ = \frac{\sqrt{3}}{2}$.

c. $\frac{1}{2} \times 13 \times h = 24.25$, so $h =$ **3.73 cm**. Use area $= \frac{1}{2} \times$ base $\times$ height with the 13 cm side as the base.

Answer to Question 19

a. Angle $SAB = 90 - 30 = 60^\circ$. At B the direction to A is 270° and to S is 310°, so angle $SBA = 310 - 270 = 40^\circ$. Bearings are measured clockwise from north, and AB points due east (090°), so angle $SAB = 90^\circ - 30^\circ$.

b. $180 - 60 - 40 = 80^\circ$. The angles of a triangle add up to 180°.

c. $AS = \frac{10 \sin 40^\circ}{\sin 80^\circ} =$ **6.53 km**. AS is opposite the 40° angle at B, and $AB = 10$ km is opposite the 80° angle at S.

d. $6.53 \sin 60^\circ =$ **5.65 km**. The distance north is the side opposite the 60° angle in the right-angled triangle with hypotenuse AS.

Answer to Question 20

a. In the right-angled triangle BCD, $\sin B = \frac{CD}{a}$, so $CD = a \sin B$.

b. Area $= \frac{1}{2} \times AB \times CD = \frac{1}{2} \times c \times a \sin B = \frac{1}{2}ac \sin B$.

c. $\frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$. Divide by $\frac{1}{2}c$: $a \sin B = b \sin A$. Divide by $\sin A \sin B$: $\frac{a}{\sin A} = \frac{b}{\sin B}$. This is the sine rule.