Lesson - Similarity and congruence

Year: Year 11

Congruence and basic similarity are on both tiers. Using the scale factor for areas and volumes is Higher tier only (marked Higher).

Starter activity

Take two pieces of card and cut out a triangle from each with sides 6 cm, 8 cm and 10 cm. Place them on top of each other. Now cut a third triangle with sides 3 cm, 4 cm and 5 cm. Does it fit on the first? How do its angles compare? How do its sides compare? Describe in your own words what is the same and what is different about the third triangle.

Congruent and similar shapes

Definition: congruent and similar

Two shapes are **congruent** if they are exactly the same shape and size (one can be moved onto the other by a reflection, rotation or translation). Two shapes are **similar** if they have the same shape but possibly a different size: one is an enlargement of the other.

Congruent triangles: the four conditions

Rule: the congruence conditions

Two triangles are congruent if any *one* of these is true:

- **SSS**: all three sides are equal.
- **SAS**: two sides and the *included* angle (the angle between them) are equal.
- **ASA** (or AAS): two angles and a corresponding side are equal.
- **RHS**: a right angle, the hypotenuse and one other side are equal.

Watch out: AAA and SSA

AAA is *not* a test for congruence: three equal angles only show the triangles are similar. SSA (two sides and a non-included angle) is also not enough.

Method: proving triangles congruent

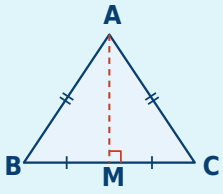
1. Write a short chain of reasons, each in its own line.
2. Match the sides and angles, using given facts, midpoints, parallel lines and common sides.

3. Finish with the rule used (SSS, SAS, ASA or RHS) and what follows.

Worked example 1: isosceles triangle proof

ABC is isosceles with $AB = AC$. M is the midpoint of BC. Prove that triangles ABM and ACM are congruent.

$AB = AC$ (given). $BM = CM$ (M is the midpoint). AM is common to both triangles. So the triangles are congruent by SSS. It follows that angle B equals angle C, and angle AMB = angle AMC. These two angles lie on a straight line so each is $90°$.



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Similar shapes

Definition: scale factor

The **scale factor** k of an enlargement is the number every length is multiplied by. A scale factor greater than 1 makes the shape bigger and between 0 and 1 makes it smaller.

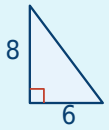
Rule: similar shapes

In similar shapes, **corresponding angles are equal** and **corresponding sides are in the same ratio**, the scale factor. To find a scale factor, divide a length in the new shape by the matching length in the old shape. Write the ratio with the larger shape on top: $\frac{\text{big}}{\text{small}} = k$. To find a side on the big shape multiply by k; on the small shape divide by k.

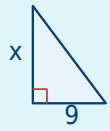
Worked example 2: a missing side

Two similar triangles have sides 6, 8 and 9, x. Find x.

$\text{scale factor} = \frac{9}{6} = 1.5$. The missing side is $8 \times 1.5 = 12$.



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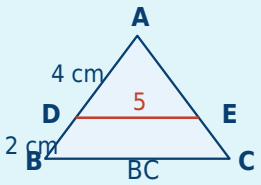


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Worked example 3: parallel lines in a triangle

Triangle ABC has $DE \parallel BC$ with D on AB and E on AC. $AD = 4$, $DB = 2$, $DE = 5$. Find BC.

Triangles ADE and ABC are similar because angle A is common and the corresponding angles at D and B (and at E and C) are equal (parallel lines). The scale factor from ADE to ABC is $\frac{AB}{AD} = \frac{6}{4} = 1.5$, so $BC = 5 \times 1.5 = 7.5$. The whole side AB is $AD + DB = 6$, not 2.



Not drawn accurately

(Higher) Areas and volumes of similar shapes

Rule (Higher): lengths, areas and volumes

If the lengths are in the ratio $1 : k$, then areas (surface areas) are in the ratio $1 : k^2$ and volumes are in the ratio $1 : k^3$. To go from a volume ratio to a length ratio take a cube root; from an area ratio take a square root. For a volume ratio of $8 : 27$ the length ratio is $2 : 3$ and the area ratio is $4 : 9$.

Worked example 4 (Higher): areas

Two similar shapes have areas 12 cm^2 and 75 cm^2. A side of the small shape is 4 cm. Find the matching side of the large shape.

The area scale factor is $\frac{75}{12} = 6.25$, so the length scale factor is $\sqrt{6.25} = 2.5$. The side becomes 10 cm.

Worked example 5 (Higher): volumes

Two similar bottles are 8 cm and 12 cm tall. The smaller holds 250 ml. How

much does the larger hold?

Length scale factor $= 1.5$, volume factor $= 1.5^3 = 3.375$. The larger bottle holds $250 \times 3.375 = 843.75 \approx 844$ ml.

Worked example 6 (Higher): statues and models

Statues of the same material have heights in the ratio $1:5$. A 2 kg statue gives what mass? A model sail at scale $1:200$ has area 0.045 m^2; find the real area.

Masses are in the ratio $1:125$ (mass is proportional to volume when the density is the same), so the larger statue has mass 250 kg. For the sail, areas scale by 200^2: $0.045 \times 200^2 = 1800$ m^2. A $1:200$ model has volumes $\dfrac{1}{8\,000\,000}$ of the real thing.

Watch out: common mistakes

- Using the length scale factor for an area or volume.
- Using AAA as a congruence test.
- Writing the triangle's vertices in the wrong order when stating similarity.
- Forgetting that the whole side includes both parts ($AB = AD + DB$).

Worked example 7: Identifying congruent triangles

Triangle ABC has $AB = 7$ cm, $BC = 9$ cm and angle $B = 40°$. Triangle DEF has $DE = 7$ cm, $EF = 9$ cm and angle $E = 40°$. Are they congruent?

Two sides and the angle *between* them are equal in the two triangles, so they are congruent by SAS. Writing the vertices in order, $ABC \equiv DEF$. If the equal angle were not between the two equal sides, there would be no guarantee (that is the ambiguous SSA case).

Worked example 8: Proving a statement with congruent triangles

In the diagram $PQ = PR$ and S is the midpoint of QR. Prove that angle PQS = angle PRS.

In triangles PQS and PRS: $PQ = PR$ (given), $QS = RS$ (S is the midpoint) and PS is common. The triangles are congruent by SSS. Corresponding angles are equal, so angle PQS = angle PRS. Each line of a proof needs a reason in brackets.

Worked example 9: Using the scale factor to find a missing length

Two similar triangles have corresponding sides 8 cm and 20 cm. A side of the smaller is 6 cm. Find the matching side of the larger.

The scale factor is $\frac{20}{8} = 2.5$. The matching side is $6 \times 2.5 = 15$ cm. Always check that the sides you divide correspond to each other: the shortest to the shortest and the longest to the longest.

Worked example 10: A scale factor less than one

Triangle A has a side of 15 cm and the similar triangle B has the matching side 6 cm. Find the scale factor from A to B and the side in B matching 10 cm in A.

The scale factor from A to B is $\frac{6}{15} = 0.4$. The side in B is $10 \times 0.4 = 4$ cm. A scale factor smaller than 1 means a reduction. Going the other way, from B to A, the scale factor is 2.5.

Worked example 11: The area of a similar shape from a ratio

Two similar triangles have heights 4 cm and 10 cm. The smaller has area 14 cm². Find the area of the larger.

The length scale factor is $\frac{10}{4} = 2.5$. The area scale factor is $2.5^2 = 6.25$. Area $= 14 \times 6.25 = 87.5$ cm^2. A frequent mistake is to multiply the area by 2.5 instead of 6.25.

Worked example 12: A volume problem with similar solids

A small bottle 10 cm tall holds 200 ml. A similar bottle is 15 cm tall. How much does it hold?

The length scale factor is 1.5, so the volume scale factor is $1.5^3 = 3.375$. Volume $= 200 \times 3.375 = 675$ ml. Notice that a bottle only 50% taller holds more than three times as much, because volume grows with the cube of the length.

Worked example 13: Finding the length scale factor from volumes

Two similar cones have volumes 40 cm^3 and 135 cm^3. Find the ratio of their heights.

The volume ratio is $40 : 135 = 8 : 27$. The length ratio is the cube root: $\sqrt[3]{8} : \sqrt[3]{27} = 2 : 3$. The ratio of heights is $2 : 3$, and the ratio of surface areas would be $4 : 9$.

Note: congruent and similar are different

Congruent shapes are exactly the same size and shape, so all their corresponding lengths and angles are equal, and one fits exactly on top of the other after a rotation,

reflection or translation. Similar shapes have the same shape but may have different sizes, so corresponding angles are equal and corresponding lengths are in the same ratio. Every pair of congruent shapes is similar (with scale factor 1), but similar shapes are not necessarily congruent.

Method: a congruence proof

1. Name the two triangles, with the vertices in corresponding order.
2. List three facts, each with a reason in brackets (given, common side, vertically opposite angles, parallel lines, radii, midpoint).
3. State the condition: SSS, SAS, ASA/AAS or RHS.
4. Write what follows: "so $AB = \ldots$" or "angle X = angle Y (corresponding parts of congruent triangles)".

Watch out: mistakes with similar shapes

- AAA proves similarity, never congruence. SSA does not prove congruence.
- Multiply by the scale factor to go to the larger shape and divide to go back.
- Areas scale by k^2 and volumes by k^3; do not use k for them.
- Match corresponding sides carefully: they are opposite equal angles.

Did you know?

Thales, the Greek mathematician who lived about 600 BC, is said to have measured the height of the Great Pyramid in Egypt using similar triangles. He waited until the length of his own shadow equalled his height. At that moment the length of the pyramid\'s shadow also equalled its height (plus half the base). The method shows how a ratio can replace a measurement that is too big to make directly.

Otto's tip

Write the scale factor first, then decide whether you need it as it is, squared or cubed. Say it in words: "lengths times 3, areas times 9, volumes times 27".