Lesson - Rates of change and areas under graphs

Year: Year 11

Real-life graphs tell a story. The steepness shows how fast something is changing, and the area underneath can show a total amount. Gradients and areas for *curved* graphs are Higher tier only (marked Higher).

Starter activity

Imagine you walk to school. Sketch a graph of your distance from home against time. What does a steep line mean? What does a flat section show? Suppose you jog the second half. Where will the line change? Now sketch a graph of your speed against time for the same journey and decide what you think the area under it represents.

Distance-time graphs

Definition: distance-time graph

On a distance-time graph, time is on the horizontal axis and distance from the start is on the vertical axis. A straight sloping line means constant speed, a horizontal line means the object is stationary and a line sloping back down means it is returning to the start. The **gradient** is the speed.

Rule: speed and average speed

$\text{speed} = \dfrac{\text{distance}}{\text{time}}$. Average speed for a whole journey is $\dfrac{\text{total distance travelled}}{\text{total time taken}}$.

Include stops in the total time.

Worked example 1: a runner

A runner travels 6 km in the first 30 minutes, rests for 10 minutes, then runs another 4 km in 20 minutes. Find the speed in the first part and the average speed.

Speed in the first part: $6 \div 0.5 = 12$ km/h. Total distance $= 10$ km, total time $= 60$ minutes $= 1$ hour. Average speed $= 10 \div 1 = 10$ km/h.

Velocity-time graphs

Definition: acceleration

On a velocity-time (speed-time) graph the gradient is the **acceleration**:

$\frac{\text{change in velocity}}{\text{time}}$, in m/s^2. A horizontal line means constant speed and a negative gradient means deceleration.

Rule: area under a velocity-time graph

The **area under the graph** is the distance travelled. Use these areas: rectangle $= bh$, triangle $= \frac{1}{2}bh$, trapezium $= \frac{1}{2}(a+b)h$, where a and b are the parallel sides.

Worked example 2: a car journey

A car accelerates from 0 to 12 m/s in 4 s, travels at 12 m/s for 10 s, then slows to rest in 6 s. Find the acceleration, the total distance and the average speed.

Acceleration $= \frac{12}{4} = 3\ \text{m/s}^2$. Distances: $\frac{1}{2} \times 4 \times 12 = 24$ m, then $12 \times 10 = 120$ m, then $\frac{1}{2} \times 6 \times 12 = 36$ m. Total $= 180$ m in 20 s, so the average speed is 9 m/s.

Other real-life graphs and units

Key fact: gradient is a rate

The gradient of any straight line graph is a rate. For a taxi, cost against distance has gradient equal to the cost per kilometre, and the y-intercept is the fixed charge. Write the units of a rate as "units of y per unit of x", for example pounds per kilometre.

Worked example 3: converting units

Convert 72 km/h into m/s, and 15 m/s into km/h.

To change km/h into m/s, multiply by 1000 and divide by 3600, which is the same as dividing by 3.6: $72\ \text{km/h} = 20\ \text{m/s}$. To go the other way, multiply by 3.6: $15\ \text{m/s} = 54\ \text{km/h}$.

(Higher) Curved graphs: gradient and area

Definition (Higher): rate of change at a point

For a curve, the gradient changes from point to point. The **rate of change at a point** is the gradient of the tangent to the curve at that point. On a distance-time graph it is the *speed at that instant*; on a velocity-time graph it is the *acceleration at that instant*.

Method (Higher): gradient of a curve

1. Place a ruler so it touches the curve at the point and does not cut it.
2. Draw the tangent line, making it long.
3. Choose two points far apart on the tangent and calculate $\dfrac{\text{change in } y}{\text{change in } x}$.

Rule (Higher): the trapezium estimate

To estimate the area under a curve, split it into strips and treat each strip as a trapezium. If the strips have width w and heights $y_0, y_1, \ldots, y_n$, the area is $\frac{1}{2}w\left(y_0 + 2y_1 + 2y_2 + \cdots + y_n\right)$. When the curve bends upwards (convex), trapezia overestimate; when it bends downwards (concave), they underestimate. More strips give a better estimate.

Worked example 4 (Higher): estimating distance

A velocity-time curve has $v = 0, 5, 8, 9$ m/s at $t = 0, 2, 4, 6$ s. Estimate the distance with trapezia.

Strip areas: $\frac{0+5}{2} \times 2 = 5$, $\frac{5+8}{2} \times 2 = 13$, $\frac{8+9}{2} \times 2 = 17$. Total $= 35$ m. The curve is bending downwards, so this is an underestimate.

Worked example 5 (Higher): different scales

1 cm across is 5 s and 1 cm up is 4 m/s. What does 3 cm^2 of area represent?

The area represents (horizontal scale) $\times$ (vertical scale) $\times$ (area in squares): $3 \times 5 \times 4 = 60$ m.

Watch out: common mistakes

- Reading a horizontal line on a distance-time graph as constant speed. It means stopped.
- Mixing minutes and hours in a speed calculation.
- Using the area under a distance-time graph. It has no meaning here.
- Drawing a tangent that crosses the curve, or using points too close together.

Worked example 6: Average speed for a two-part journey

A runner goes 6 km at 12 km/h and then 6 km at 8 km/h. Find the average speed.

Times: $\dfrac{6}{12} = 0.5$ h and $\dfrac{6}{8} = 0.75$ h. Total time 1.25 h for 12 km. Average speed $= \dfrac{12}{1.25} = 9.6$ km/h. This is not the mean of 12 and 8 (which is 10) because more time is spent at the slower speed. Always use total distance divided by total time.

Worked example 7: Reading acceleration from a velocity-time graph

A velocity-time graph goes from $(0, 5)$ to $(8, 29)$ in a straight line. Find the acceleration and the distance in the 8 seconds.

Gradient $= \dfrac{29 - 5}{8} = 3$ m/s^2. The area under the line is a trapezium: $\frac{1}{2}(5 + 29) \times 8 = 136$ m. The units of the area are metres because velocity (m/s) times time (s) gives m.

Worked example 8: Constant speed from a distance-time graph

On a distance-time graph a straight line goes through $(1, 20)$ and $(4, 50)$. What is the speed?

Gradient $= \dfrac{50 - 20}{4 - 1} = 10$. The units are those of distance divided by those of time: if the distance is in km and the time in hours the speed is 10 km/h. A steeper line means a higher speed, and a line sloping down means returning.

Worked example 9: A tangent to a curve

A distance-time curve has a tangent at $t = 3$ passing through $(1, 2)$ and $(5, 14)$. Estimate the speed at $t = 3$.

The speed at an instant is the gradient of the tangent: $\dfrac{14 - 2}{5 - 1} = 3$. Draw the tangent so that it just touches the curve at the point and use a large triangle to calculate the gradient accurately.

Worked example 10: Using the area under a graph for other quantities

A graph shows the flow rate of water in litres per minute against time in minutes. Between 0 and 10 minutes the flow is steady at 20 L/min. How much water has flowed?

Area $= 20 \times 10 = 200$ litres. The area under a rate graph gives the total amount: litres per minute multiplied by minutes. Likewise, the area under a speed-time graph gives distance and the gradient of a distance-time graph gives speed.

Worked example 11: Converting between m/s and km/h

Show that 1 m/s is 3.6 km/h and use this to convert 30 m/s.

1 m/s is 3 600 m in an hour, which is 3.6 km/h. So $30 \times 3.6 = 108$ km/h. To convert the other way, divide by 3.6: 54 km/h is $54 \div 3.6 = 15$ m/s.

Worked example 12: An estimate for unequal strips

The speed of a bike at 0, 1, 3 seconds is 0, 4, 8 m/s. Estimate the distance.

Two trapezia with different widths: $\frac{1}{2} \times 1 \times (0 + 4) = 2$ and $\frac{1}{2} \times 2 \times (4 + 8) = 12$. The total is 14 m. Each trapezium uses its own width, which is the time gap for that strip.

Note: the three key links between graphs

Remember three links. On a distance-time graph the **gradient is speed**. On a velocity-time graph the **gradient is acceleration** and the **area under the graph is distance**. For a curved graph, the gradient at a point is found with a tangent and the area is estimated with trapezia. Write the units of every answer: m/s for speed, m/s² for acceleration and m for distance. A flat line means constant value, which on a distance-time graph means stopped, and on a velocity-time graph means constant speed.

Method: estimating the gradient of a curve

1. Place a ruler so that it just touches the curve at the point and appears to follow its direction.
2. Draw a long straight line through the point.
3. Choose two points far apart on the tangent and draw a right-angled triangle.
4. Work out the vertical change divided by the horizontal change, and include the units.

Watch out: common mistakes with rate graphs

- Mixing units: if time is in minutes and distance in km, the speed is km per minute, not km/h.
- Confusing a horizontal line on a distance-time graph (stopped) with one on a velocity-time graph (steady speed).
- Using the mean of speeds as the average speed instead of total distance divided by total time.
- Forgetting that area under a velocity-time graph is distance only when velocity and time units match.

Did you know?

Galileo Galilei, an Italian scientist who lived about 400 years ago, rolled balls down slopes to study acceleration. He found that the distance travelled grows with the square of the time, so a graph of distance against time is a curve. Modern speed cameras and fitness watches use the same idea: they measure position at regular times and calculate speed and acceleration from the gradient of the graph.

Otto's tip

Write the units next to every number. If your answer for acceleration comes out in metres, something has gone wrong. Gradient of distance gives speed, gradient of speed gives acceleration, and area under speed gives distance.