Lesson - Pythagoras and trigonometry in right-angled triangles

Year: Year 11

Right-angled triangles are everywhere in GCSE maths. This topic is on both tiers; three-dimensional problems are mostly Higher. Always draw a clear sketch and label the sides before you start.

Starter activity

Stand outside, or look at a picture of a tall building. How could you find its height without climbing it? Suppose you walk 20 m away and measure the angle from the ground to the top using a protractor and a plumb line. Which pieces of information have you got, and which right-angled triangle can you draw? Which of the three ratios sine, cosine or tangent do you need?

Pythagoras' theorem

Theorem: Pythagoras

In a right-angled triangle the square on the hypotenuse (the longest side, opposite the right angle) equals the sum of the squares on the other two sides:

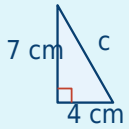
$$a^2 + b^2 = c^2$$

- To find the **hypotenuse**: add the squares, then square root.
- To find a **shorter side**: subtract the squares, then square root.

Worked example 1: the hypotenuse

Legs 4 cm and 7 cm. Find the hypotenuse.

$c^2 = 16 + 49 = 65$, so $c = \sqrt{65} = 8.06$ cm (2 d.p.).



Not drawn accurately

Worked example 2: a ladder

A 5 m ladder has its foot 1.4 m from a wall. How high does it reach?

Height $= \sqrt{5^2 - 1.4^2} = \sqrt{25 - 1.96} = \sqrt{23.04} = 4.8$ m.



Not drawn accurately

Key fact: triples, converse and distance

Pythagorean triples to remember: $3, 4, 5$; $5, 12, 13$; $8, 15, 17$; $7, 24, 25$. Multiples such as $6, 8, 10$ also work. If $a^2 + b^2 = c^2$ for a triangle's sides, the triangle is right-angled. The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Trigonometry: labelling and ratios

Definition: hypotenuse, opposite and adjacent

Label the sides relative to the angle you are using: **hypotenuse** (opposite the right angle), **opposite** (across from the angle) and **adjacent** (next to the angle, not the hypotenuse).

Rule: SOH CAH TOA

$$\sin\theta = \frac{O}{H}, \qquad \cos\theta = \frac{A}{H}, \qquad \tan\theta = \frac{O}{A}$$

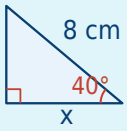
Method: finding a side

1. Label the three sides.
2. Choose the ratio that uses the side you know and the side you want.
3. Write the equation and rearrange. If the unknown is on the bottom, for example $\sin 35^\circ = \dfrac{7}{x}$, rearrange to $x = \dfrac{7}{\sin 35^\circ}$.

Worked example 3: finding a side

Angle 40°, hypotenuse 8 cm. Find the adjacent side.

$\cos 40^\circ = \dfrac{x}{8}$, so $x = 8\cos 40^\circ = 6.13$ cm.



Not drawn accurately

Worked example 4: finding an angle

Opposite 7, adjacent 9. Find the angle.

Use the inverse function. $\tan\theta = \dfrac{7}{9}$, so $\theta = \tan^{-1}\dfrac{7}{9} = 37.9^\circ$. Check that your calculator is in **degree mode**.

Definition: angles of elevation and depression

The **angle of elevation** is measured upwards from the horizontal; the **angle of depression** is measured downwards from the horizontal. The two are equal between the same two points because of alternate angles. A tree seen at 32° from 20 m away has height $20\tan 32^\circ = 12.5$ m.

Exact trigonometric values

Key fact: exact values

You must know these without a calculator.

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Worked example 5: exact values and an isosceles triangle

(a) A right-angled triangle has hypotenuse 12 and an angle of 60°: find the other sides exactly. (b) Find the height, area and base angle of an isosceles triangle with sides $10, 10, 12$.

(a) Adjacent $= 12\cos 60^\circ = 6$ and opposite $= 12\sin 60^\circ = 6\sqrt{3}$. (b) Split it down the middle: half the base is 6, height $= \sqrt{100 - 36} = 8$, area $= \frac{1}{2} \times 12 \times 8 = 48$ and each base angle $= \cos^{-1}\dfrac{6}{10} = 53.1^\circ$.

(Higher) Three-dimensional problems

Method (Higher): 3D problems

1. Sketch the solid and mark the line or angle you need.
2. Find the right-angled triangle you need, often more than one in a row.
3. Use Pythagoras or trigonometry in each triangle in turn, keeping full calculator values.

Worked example 6 (Higher): cuboid and pyramid

Find the space diagonal of a cuboid $3 \times 4 \times 12$ and the angle it makes with the base. For a square-based pyramid with base 8 and height 6, find the slant height and a sloping edge.

The diagonal across the base is $\sqrt{3^2 + 4^2} = 5$. The space diagonal uses a second right-angled triangle with sides 5 and 12: $\sqrt{5^2 + 12^2} = 13$. The angle with the base is $\tan^{-1}\dfrac{12}{5} = 67.4^\circ$. For the pyramid, the slant height of a face is $\sqrt{6^2 + 4^2} = \sqrt{52} = 7.21$ and a sloping edge is $\sqrt{6^2 + (4\sqrt{2})^2} = \sqrt{68} = 8.25$.

Problems with several steps

Worked example 7: distance and bearing

A ship leaves port and sails 16 km due east, then 12 km due north. How far is it from the port and on what bearing?

The two legs form a right angle, so the distance is $\sqrt{16^2 + 12^2} = \sqrt{400} = 20$ km. The angle at the port between east and the direct line is $\tan^{-1}\dfrac{12}{16} = 36.9^\circ$. A bearing is measured clockwise from north, so the bearing is $90^\circ - 36.9^\circ = 053.1^\circ$.

Note: choosing the best method

If you know two sides and want the third, use Pythagoras. If you know an angle and a side, or two sides and want an angle, use trigonometry. If a question says "give your answer as a surd" or "do not use a calculator", use Pythagoras with a simplified root or an exact trig value: $\sqrt{50} = 5\sqrt{2}$ and $\sqrt{12} = 2\sqrt{3}$. Simplify a surd by taking out the largest square number factor. Draw the triangle yourself even when a diagram is given, keep the same units, and write the unrounded value ($8\cos 40^\circ = 6.1283\ldots$) before the rounded one.

Watch out: common mistakes

- Adding the squares when you should subtract (finding a short side).
- Calculator in radians or gradians mode.
- Labelling opposite and adjacent from the wrong angle.
- Rounding too soon: keep full calculator values until the end.
- Using sine, cosine or tangent in a triangle without a right angle.

Worked example 8: Pythagoras in an isosceles triangle

An isosceles triangle has base 10 cm and equal sides 13 cm. Find its height and area.

The height meets the base at its midpoint, forming a right-angled triangle with hypotenuse 13 and base 5. Height $= \sqrt{13^2 - 5^2} = 12$ cm. Area $= \frac{1}{2} \times 10 \times 12 = 60$ cm^2. Dropping a perpendicular height is the standard way to use Pythagoras in isosceles and equilateral triangles.

Worked example 9: Choosing the correct trigonometric ratio

In a right-angled triangle the side opposite angle x is 6 cm and the hypotenuse is 10 cm. Find x.

Label the sides: opposite is 6 and hypotenuse is 10. These two appear in the sine ratio (SOH). $\sin x = \dfrac{6}{10} = 0.6$, so $x = \sin^{-1} 0.6 = 36.9^\circ$. If you know the opposite and the adjacent use TOA, and if you know the adjacent and the hypotenuse use CAH.

Worked example 10: Angle of depression

From the top of a cliff 40 m high, the angle of depression of a boat is 22°. How far is the boat from the foot of the cliff?

The angle of elevation of the top from the boat equals the angle of depression, 22°. Distance $= \dfrac{40}{\tan 22^\circ} = \dfrac{40}{0.4040} = 99.0$ m. Always draw the horizontal line first and measure the depression angle *downwards* from it.

Worked example 11: Distance between two points

Find the distance between $A(1, 2)$ and $B(7, 10)$.

Make a right-angled triangle with horizontal side $7 - 1 = 6$ and vertical side $10 - 2 = 8$. $AB = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$. This is the distance formula, and it is just Pythagoras\' theorem applied to the coordinate grid.

Worked example 12: Two triangles in one diagram

In the diagram $AB = 12$ cm, angle $BAC = 90°$, angle $ABC = 35°$ and D is on BC with $AD \perp BC$. Find AD.

First find AC in triangle ABC: $AC = 12 \tan 35° = 8.40$ cm. Then use triangle ABD (right-angled at D): $AD = AB \sin 35° = 12 \times 0.5736 = 6.88$ cm. Choose the triangle that contains the side you want and two pieces of information.

Worked example 13: A 3D problem (Higher)

A vertical pole 6 m tall stands at the corner A of a rectangular field 8 m by 15 m. Find the angle between the line from the top of the pole to the opposite corner and the ground.

The diagonal of the field is $\sqrt{8^2 + 15^2} = 17$ m. The angle is $\tan^{-1} \dfrac{6}{17} = 19.4°$. In 3D problems always find the length in the horizontal plane first, then use a right-angled triangle that is vertical.

Worked example 14: Pythagoras to check a diagram

A rectangular gate is 1.2 m wide and 0.9 m tall. How long is the diagonal brace?

The brace is the hypotenuse: $\sqrt{1.2^2 + 0.9^2} = \sqrt{1.44 + 0.81} = \sqrt{2.25} = 1.5$ m. The sides 0.9, 1.2 and 1.5 are a 3, 4, 5 triangle scaled by 0.3.

Note: choosing the best method

If a right-angled triangle has **two sides known** and you want the third side, use Pythagoras. If you want an angle, or you know one side and one angle, use trigonometry. Label the sides opposite, adjacent and hypotenuse relative to the angle you are using before you write any ratio. Check the answer for sense: the hypotenuse must be the longest side and an acute angle must be less than $90°$. Set your calculator to degrees and use the inverse button for angles.

Method: a trigonometry question in four steps

1. Draw the triangle and label the sides O, A and H relative to the angle.
2. Choose the ratio that uses the two sides involved (SOH, CAH or TOA).
3. Write the equation, then rearrange: multiply to find a numerator, divide to find a denominator, use the inverse function to find an angle.
4. Round at the end and give units.

Watch out: common mistakes with right-angled triangles

- Adding the squares when you want a shorter side: you must **subtract**.
- Using the wrong side as the hypotenuse; it is the longest side, opposite the right angle.
- Calculator in radians mode, giving silly angles.
- Rounding in the middle of a multi-step problem, which creates errors in the last digit.

Did you know?

The Egyptian surveyors who rebuilt field boundaries after each flood of the Nile used a rope with 12 equally spaced knots. Stretched into a triangle with sides 3, 4 and 5 units it gave an exact right angle. The theorem is named after the Greek Pythagoras, who lived about 2 500 years ago, but the Babylonians knew it more than a thousand years earlier, and a clay tablet called Plimpton 322 lists whole-number triples.

Otto's tip

Check every answer for sense. The hypotenuse must be the longest side. If your "short" side comes out longer than the hypotenuse, you added when you should have subtracted.