Lesson - Problem solving and GCSE exam practice

Year: Year 11

Every GCSE maths paper tests three assessment objectives. This chapter gives you a method for the problem-solving questions, which separate the higher grades, and mixed exam-style questions covering the whole course.

Starter activity

Look at this problem: "A shop sells pens in packs of 6 and in packs of 10. Tina buys some packs and gets exactly 46 pens. How many packs of each type might she have bought?" Before you use any algebra, try some values of your own. How many different answers can you find? What does this tell you about when a problem has a single answer and when it has several?

The assessment objectives and the papers

Definition: AO1, AO2 and AO3

AO1 (about 50% of marks) is using and applying standard techniques. **AO2** (about 25%) is reasoning, interpreting and communicating mathematically. **AO3** (about 25%) is solving problems in mathematical and non-mathematical contexts.

Key fact: the three GCSE papers

Paper 1 is non-calculator, and Papers 2 and 3 allow a calculator. Each paper is 1 hour 30 minutes and has 80 marks, so you have a little over a minute per mark. Questions get harder towards the end of each paper. On Paper 1 you must know exact trig values, surds, fractions and non-calculator methods.

A problem-solving method

Method: READ, PLAN, DO, CHECK

1. **Read** the question twice. Underline the numbers and the command word. What am I being asked to find?
2. **Plan** your steps. Is it a ratio, algebra or area question? Write down which formulas you need. Most multi-step questions have three to five steps.
3. **Do** each step in order, showing your working. Each step usually earns a mark.
4. **Check** your answer: units, rounding, sense. Substitute back. Re-read the question to make sure you answered it.

Definition: command words

Word	What to do
Work out / Calculate	Show your method and give the answer
Show that	Prove the given result with every step; do not just verify the number
Prove	Use algebra or logic to show it is true for all cases
Explain / Give a reason	Write a sentence using the mathematical fact
Estimate	Round numbers to 1 significant figure first
Give your answer to ...	Round at the end to the stated accuracy

Worked problems

Worked example 1: a multi-step percentage problem

A coat costs £80. In a sale the price is reduced by 15%. In a second sale the new price is reduced by a further 10%. Find the final price.

Multiplier for 15% off is 0.85 and for 10% off is 0.9. $80 \times 0.85 \times 0.9 = 61.20$. The final price is £61.20. (This is not 25% off, which would be £60.)

Worked example 2: algebra in context

A rectangle has length $2x + 3$ and width $x - 1$. The perimeter is 34. Find the area.

$2(2x + 3) + 2(x - 1) = 34 \Rightarrow 6x + 4 = 34 \Rightarrow x = 5$. Length $= 13$, width $= 4$, area $= 52$.

Worked example 3: a path around a field

A rectangular field is 40 m by 30 m. A path 2 m wide runs around the inside edge. Find the area of the path.

The inner rectangle is $(40 - 4)$ by $(30 - 4)$, which is 36 by $26 = 936$. The field is 1200. Path $= 1200 - 936 = 264$ m^2.

Rule: proof by algebra

To prove a statement for all integers, use letters: n for any integer, $n + 1$ for the next, $2n$ for an even number and $2n + 1$ for an odd number. Finish by saying the result is a multiple of what you want.

Worked example 4: proof

Prove that the sum of three consecutive integers is a multiple of 3.

Let the integers be n, $n+1$, $n+2$. Sum $= 3n+3 = 3(n+1)$, which is 3 times an integer.

Strategies and checklist

Key fact: when you are stuck

- Draw a diagram and label everything. Many geometry questions have no diagram.
- Try a simple number to see what is happening.
- Write down everything you can find from the information. Method marks are awarded for correct steps.
- Work backwards from the answer when the question is a "show that".
- Move on if you spend more than two minutes with no progress, and come back.

Note: topic checklist

Before the exam, check that you can do all of these without notes: fractions, decimals and percentages (including reverse and compound percentages); ratio and proportion; solving equations and inequalities; factorising and quadratics; simultaneous equations; sequences; gradients and graphs; Pythagoras and trigonometry; angle facts and polygons; area, volume, circles and sectors; transformations; probability with tree diagrams; averages, charts and scatter graphs. Higher students should add surds, functions, circle theorems, vectors and iteration.

Note: presentation

Write neatly and in order. Put your final answer on the answer line with units. Do not cross out working unless you replace it. Round only at the end. If a question says "Give your answer to 3 significant figures", that is a mark in its own right.

Watch out: common mistakes

- Not reading the question: finding the perimeter when asked for the area.
- Rounding too soon in multi-step questions.
- Forgetting units or leaving the answer in the wrong units.
- Giving an answer without working in a "show that" question.
- Leaving a question blank. A guess or a first step can earn a mark.

Worked example 5: Read, plan, do, check

A rectangle has perimeter 34 cm and length 5 cm more than its width. Find its area.

Read: unknowns are length and width. Plan: let the width be w, so the length is $w+5$

and $2(2w + 5) = 34$. Do: $4w + 10 = 34$, $4w = 24$, $w = 6$ and length $= 11$. Area $= 66$ cm ^{2}. Check: perimeter $= 2 \times 17 = 34$ ✓. Writing the four stages shows the examiner your method and earns marks even if you slip.

Worked example 6: Using command words

What is the difference between "show that", "work out" and "explain" in an exam question?

"Work out" or "calculate" asks for a numerical answer with working. "Show that" gives you the answer and you must prove it by writing every step. "Explain" or "give a reason" needs a sentence, not only a number. "Hence" means use the previous result. Underline the command word before you start; many marks are lost by answering a different question from the one asked.

Worked example 7: A multi-step percentage problem

A shop sells a laptop for £540, which includes 20% VAT. Find the price before VAT and the amount of VAT.

The price including VAT is 120% of the original, so the multiplier is 1.2. Price before VAT $= 540 \div 1.2 = 450$. VAT $= 540 - 450 = 90$. A common error is to take 20% off £540, which gives £432. Always divide by the multiplier in a reverse problem.

Worked example 8: Working with ratio and algebra together

The ages of Ali and Ben are in the ratio $3 : 5$. In 6 years the ratio will be $2 : 3$. Find their ages now.

Let the ages now be $3k$ and $5k$. In 6 years: $\dfrac{3k + 6}{5k + 6} = \dfrac{2}{3}$, so $3(3k + 6) = 2(5k + 6)$, $9k + 18 = 10k + 12$, so $k = 6$. Ali is 18 and Ben is 30. Check: in 6 years they are 24 and 36, and $24 : 36 = 2 : 3$ ✓.

Worked example 9: Proof by algebra

Prove that the difference between the squares of two consecutive numbers is always odd.

Let the numbers be n and $n + 1$. $(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$. Since $2n$ is even, $2n + 1$ is odd. The proof works for every whole number n, which is what makes it a proof rather than an example.

Worked example 10: Choosing between methods

A right-angled triangle has hypotenuse 13 cm and one side 5 cm. Find the other side and the smaller angle.

Two sides are known and there is a right angle, so use Pythagoras: $\sqrt{13^2 - 5^2} = 12$ cm. For the smaller angle, opposite the 5 cm side, $\sin x = \frac{5}{13}$, so $x = 22.6°$. When you have a right angle and two sides, think first Pythagoras for a missing length and trigonometry for an angle.

Worked example 11: Estimating to check an answer

Estimate $\frac{4.9 \times 31}{0.52}$ and explain why it is useful.

Round each number to one significant figure: $\frac{5 \times 30}{0.5} = 300$. A calculator gives about 292. The estimate shows that the answer is sensible. If you had pressed a wrong key and got 29.2 or 2920, the estimate would warn you.

Note: the three GCSE assessment objectives

Examiners award marks for three skills. **AO1** is using and applying standard techniques, such as solving an equation. **AO2** is reasoning, interpreting and communicating mathematically, such as proof and giving explanations. **AO3** is solving problems in context, where you choose the method yourself. Roughly half of the marks in the papers are for AO2 and AO3 together, so practise unfamiliar problems and not only routine skills. Always write a final sentence with units that answers the question in context.

Method: when you are stuck

1. Draw a diagram or table, and label every known quantity.
2. Write down every formula that might apply (Pythagoras, area, ratio, averages) and see which one uses what you know.
3. Try a simple number to see what is happening.
4. Work backwards from what you want to find, and write something for every part: method marks are available for partial working.

Watch out: common mistakes in exam papers

- Not reading the whole question, for example forgetting to round or to give units.
- Rounding too early; keep full calculator values and round at the end.
- Giving only a number when asked to "show that" or "explain".
- Leaving a question blank. Always attempt every part, because a partial method scores marks.

Did you know?

The mathematician George Pólya wrote a famous book in 1945 called *How to Solve It*. It proposed four steps: understand the problem, make a plan, carry it out and look back. The same four steps, here called read, plan, do and check, are used by mathematics teachers around the world. They are also used by engineers and scientists whenever they meet a new and unfamiliar problem.

Otto's tip

When time is short, secure the marks on the questions you can do, then go back. Every method mark counts, so never leave a blank.