Lesson - Iteration, growth and decay

Year: Year 11

Compound interest, growth and decay appear on both papers. Solving equations by **iteration** is Higher tier only (marked Higher). Both halves repeat a process, and a calculator makes that quick if you use the answer key.

Starter activity

Imagine you have £100 in a bank account paying 10% interest at the end of every year. How much do you have after 1 year? After 2 years? Do you earn the same amount of interest in the second year as in the first? Compare this with an account that pays a fixed £10 each year. Which is better after 10 years? Try to explain in one sentence why the first account grows faster.

Percentage multipliers

Definition: percentage multiplier

To increase an amount by a percentage, multiply by $1 + \frac{\text{percent}}{100}$. To decrease, multiply by $1 - \frac{\text{percent}}{100}$.

Change	Multiplier
increase by 6%	1.06
increase by 2.5%	1.025
decrease by 15%	0.85
decrease by 0.5%	0.995

Compound interest and exponential growth

Definition: compound interest

With **compound interest** the interest is added each year and the next year's interest is worked out on the bigger total. Simple interest pays the same amount every year, based on the original amount only.

Rule: compound growth and decay

If the starting amount is P, the yearly multiplier is r and the time is n years, the final amount is

$$A = P \times r^{n}$$

The same formula models population growth ($r > 1$) and depreciation or decay ($r < 1$). In $N = A \times r^t$, A is the starting amount and r the multiplier per step; in $N = 50 \times 2^t$ the number doubles each step.

Worked example 1: compound interest

£500 is invested for 3 years at 4% a year compound interest. Find the final value and the interest.

Multiplier $= 1.04$. $A = 500 \times 1.04^3 = 500 \times 1.124864 = 562.432$. The value is £562.43 and the interest is $562.43 - 500$ =£62.43.

Worked example 2: depreciation

A car worth £12 000 loses 20% of its value each year. Find its value after 3 years.

Multiplier $= 0.8$. $12000 \times 0.8^3 = 12000 \times 0.512$ =£6144.

Worked example 3: different rates each year

£800 earns 3% in year 1 and 2% in year 2. Find the final value.

Multiply the separate multipliers: $800 \times 1.03 \times 1.02$ =£840.48.

Reverse problems and comparisons

Method: reverse compound problems

1. To find the original amount, divide the final amount by r^n.
2. To find a rate, work out the overall multiplier and take the nth root.
3. To find the number of years, try values of n until the target is passed.

Worked example 4: working backwards

A sum grows by 10% a year for two years to £1210. Find the original. £5000 grows to £5618 in two years: find the rate. How many years for £1000 at 5% to pass £1200?

Original $= \dfrac{1210}{1.1^2} = \dfrac{1210}{1.21}$ =£1000. For the rate, $r^2 = \dfrac{5618}{5000} = 1.1236$, so $r = 1.06$, a rate of 6%. For the years: $1.05^3 = 1.1576$ gives £1157.63, not enough; $1.05^4 = 1.2155$ gives £1215.51, so it takes 4 years.

Key fact: simple against compound

£1000 at 5% simple interest earns £50 a year, so after 10 years it is worth £1500. With compound interest it is worth 1000×1.05^{10} =£1628.89. Compound interest grows faster because interest earns interest.

Worked example 5: "is she correct?"

Someone says 5% a year for 2 years is the same as 10%. Is that right?

No: $1.05^2 = 1.1025$, a 10.25% increase. A decrease of 10% followed by an increase of 10% has multiplier $0.9 \times 1.1 = 0.99$, so the quantity ends 1% lower.

(Higher) Iteration

Definition (Higher): iteration

Some equations cannot be solved by algebra. Rearrange them into the form $x = g(x)$ and build a sequence $x_{n+1} = g(x_n)$ from a starting value x_0. If the values settle towards a number, that number is a solution.

Method (Higher): using iteration

1. Show a root exists: a change of sign of $f(x)$ between two values means a root lies between them.
2. Rearrange to $x = g(x)$ and enter the starting value.
3. Press equals, then use the **Ans** key in place of x_n and press equals again and again.
4. Write every value without rounding early, and state the answer to the accuracy asked.

Worked example 6 (Higher): iterating

$x_{n+1} = \sqrt{x_n + 2}$ with $x_0 = 1$. Find x_1, x_2, x_3.

$x_1 = \sqrt{3} = 1.7321$. $x_2 = \sqrt{3.7321} = 1.9319$. $x_3 = \sqrt{3.9319} = 1.9829$. The values are heading towards 2, which solves $x^2 = x + 2$ (that is $x^2 - x - 2 = 0$).

Worked example 7 (Higher): showing a root exists

Show that $x^3 + x - 3 = 0$ has a solution between 1 and 2, then find it to 2 decimal places.

Let $f(x) = x^3 + x - 3$. $f(1) = -1$ (negative) and $f(2) = 7$ (positive). The sign change shows a root between 1 and 2, because the graph is continuous. Rearranging gives $x^3 = 3 - x$, so $x_{n+1} = \sqrt[3]{3 - x_n}$. With $x_0 = 1.2$: $x_1 = 1.2164$, $x_2 = 1.2127$, $x_3 = 1.2136$. To 2 decimal places the root is 1.21.

Watch out: common mistakes

- Using 0.2 instead of 0.8 for a 20% decrease.
- Multiplying the rate by the number of years instead of using a power.
- Rounding the multiplier, for example writing 1.12 for 1.1236.
- Writing money to the wrong number of decimal places; give pounds to 2 d.p.

Worked example 8: Increasing by a percentage repeatedly

A town of 25 000 people grows by 3% each year. Find the population after 4 years, to the nearest 100.

The multiplier is 1.03. After 4 years the population is $25\,000 \times 1.03^4 = 25\,000 \times 1.1255 = 28\,138$. To the nearest 100 that is 28 100. Note that this is more than $25\,000 + 4 \times 3\%$ of $25\,000 = 28\,000$, because the percentage is applied to a bigger number each year.

Worked example 9: Finding the original value after a decrease

A television is on sale at £420 after a discount of 30%. What was the original price?

The sale price is 70% of the original, so the multiplier is 0.7. Original price $= 420 \div 0.7 = 600$. A common mistake is to find 30% of 420 and add it on; this gives the wrong answer because the 30% was taken from the original, not from the sale price.

Worked example 10: Equating a decay to a target

A radioactive sample falls by 10% each day. Starting with 200 g, after how many days is less than 100 g left?

Use 200×0.9^n. After 6 days: $200 \times 0.531 = 106.3$ g. After 7 days: $200 \times 0.478 = 95.7$ g. So it takes 7 days. This is a trial-and-improvement method: the mass first drops below half after 7 days.

Worked example 11: Using the multiplier for two different changes

A house price rises by 8% in the first year and then falls by 5% in the second year. By what overall percentage does the price change?

The multipliers are 1.08 and 0.95. The overall multiplier is $1.08 \times 0.95 = 1.026$. This is an increase of 2.6%. Do not simply subtract 5 from 8, because the 5% is taken from the new, higher price.

Worked example 12: An iterative formula for a solution

Show that $x^3 - 5x + 1 = 0$ can be rearranged to $x = \dfrac{x^3 + 1}{5}$ and use $x_0 = 0$ to find a root to 3 decimal places.

Rearrange: $5x = x^3 + 1$, so $x = \dfrac{x^3 + 1}{5}$. Then $x_1 = \dfrac{1}{5} = 0.2$, $x_2 = \dfrac{0.008 + 1}{5} = 0.2016$, $x_3 = 0.2016$ (to 4 d.p.: 0.20164). The root is $x = 0.202$ to 3 decimal places. Check: $0.202^3 - 5 \times 0.202 + 1 = 0.008 - 1.01 + 1 = -0.002 \approx 0$.

Worked example 13: A compound growth formula for exam questions

Write a formula for the value V of £P after n years at $r\%$ compound growth, and use it for £800 at 3.5% for 6 years.

$V = P\left(1 + \dfrac{r}{100}\right)^n$. Substituting $P = 800$, $r = 3.5$ and $n = 6$ gives $V = 800 \times 1.035^6 = 800 \times 1.2293 = 983.40$. For decay replace the plus with a minus: $V = P\left(1 - \dfrac{r}{100}\right)^n$.

Worked example 14: Percentage change from start to end

An investment grows from £2 500 to £3 000. Find the percentage increase and the yearly rate if this happened over 3 years.

Percentage increase $= \dfrac{500}{2500} \times 100 = 20\%$. The overall multiplier is 1.2. The yearly rate satisfies $m^3 = 1.2$, so $m = \sqrt[3]{1.2} = 1.0627$: about 6.3% per year. This is a little less than $20 \div 3 = 6.7\%$ because of compounding.

Method: compound percentage problems

1. Change each percentage to a multiplier: add the percentage to 1 for an increase and subtract it from 1 for a decrease.
2. For repeated changes raise the multiplier to the power of the number of periods, or multiply different multipliers together.
3. For a reverse problem, **divide** by the multiplier.
4. Round only at the end, then write the answer with correct units and a sensible degree of accuracy.

Watch out: percentage traps

- Three years of 20% depreciation is **not** a 60% fall; it is $1 - 0.8^3 = 48.8\%$.
- An increase of 10% followed by a decrease of 10% is not zero change, because the decrease is applied to a bigger amount.
- In reverse problems you divide by the multiplier; you do not subtract the percentage from the final amount.

- For iteration, use the full calculator value in each step, and round only when you write the answer.

Did you know?

The mathematician Jacob Bernoulli, who lived in Switzerland in the 17th century, discovered the number $e = 2.718\ldots$ while studying compound interest. He asked what happens if interest is paid more and more often during the year. The amount of money does not grow without limit but settles towards e times the starting amount when the rate is 100% for one year. This number is now used for population growth, radioactive decay and many other real situations.

Otto's tip

Ask yourself if the answer makes sense. Growth must give a bigger number, decay a smaller one. And for iteration, say what you found: "the values are tending to 1.21, so the solution is 1.21 to 2 decimal places".