Lesson - Functions and graph transformations

Year: Year 11

This topic is Higher tier only. It tests function notation, composite functions, inverses and how a graph moves. Every mark comes from careful, tidy algebra, so write each step down.

Starter activity

A machine takes a number, doubles it and adds 3. Another machine takes a number and squares it. If you put 4 into the first machine and then put the result into the second, what do you get? What if you use the machines in the opposite order? Are the answers the same? Now try to design a third machine that undoes the first machine. What does it do to a number, and in what order?

Function notation

Definition (Higher): function

A **function** is a rule that turns an input into exactly one output. We write $f(x) = 3x + 2$ and read it as "f of x equals 3x plus 2". The bracket holds the input.

Worked example 1: using a function

$f(x) = 3x + 2$. Find $f(4)$ and solve $f(x) = 20$.

Replace every x with 4: $f(4) = 3 \times 4 + 2 = 14$. For $f(x) = 20$, $3x + 2 = 20$ gives $x = 6$.

Composite functions

Definition (Higher): composite function

$fg(x)$ means "do g first, then f". Write it as $f(g(x))$. The order matters, so fg and gf are usually different.

Worked example 2: fg and gf

$f(x) = x^2 + 1$ and $g(x) = x - 3$. Find $fg(x)$ and $gf(x)$.

$fg(x) = (x - 3)^2 + 1 = x^2 - 6x + 10$. $gf(x) = (x^2 + 1) - 3 = x^2 - 2$.

Worked example 3: solving a composite equation

$f(x) = 2x - 1$ and $g(x) = x^2 + 3$. Solve $gf(x) = 7$.

$gf(x) = (2x-1)^2 + 3 = 4x^2 - 4x + 4$. Set equal to 7: $4x^2 - 4x - 3 = 0$, so $(2x-3)(2x+1) = 0$ and $x = \frac{3}{2}$ or $x = -\frac{1}{2}$.

Inverse functions

Method (Higher): finding an inverse function

1. Write $y = f(x)$.
2. Make x the subject.
3. Replace y with x in the answer to get $f^{-1}(x)$.
4. Check: $f^{-1}(f(x)) = x$ always.

Worked example 4: two inverses

Find the inverse of $f(x) = 5x - 3$ and of $f(x) = \dfrac{3}{x-1}$.

For $y = 5x - 3$: $x = \dfrac{y+3}{5}$, so $f^{-1}(x) = \dfrac{x+3}{5}$. For $y = \dfrac{3}{x-1}$: $x - 1 = \dfrac{3}{y}$, so $x = \dfrac{3}{y} + 1$ and $f^{-1}(x) = \dfrac{3}{x} + 1$ $(x \neq 0)$.

Domain and range

Definition (Higher): domain and range

The **domain** is the set of inputs that a function can accept. The **range** is the set of outputs it produces. For $f(x) = \dfrac{1}{x-2}$ the domain excludes $x = 2$ because you cannot divide by zero. For $f(x) = x^2$ the range is $f(x) \geq 0$. For $f(x) = 2^x$ the range is $f(x) > 0$. Look for a denominator that could be zero.

Transformations of graphs

Rule (Higher): transformations of $y = f(x)$

New equation	Transformation
$y = f(x) + a$	translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$ (up a)
$y = f(x + a)$	translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ (left a)

$y = -f(x)$	reflection in the x-axis
$y = f(-x)$	reflection in the y-axis

A change *inside* the bracket affects x and acts "backwards": $f(x+3)$ moves the graph 3 to the *left*. A change *outside* the bracket affects y and does what you expect.

Worked example 5: moving a minimum

The graph of $y = f(x)$ has a minimum at $(1, -2)$. Find the turning points of $y = f(x+2)$, $y = f(x) - 5$ and $y = -f(x)$.

The minimum of $y = f(x+2)$ is at $(-1, -2)$. The minimum of $y = f(x) - 5$ is at $(1, -7)$. The graph of $y = -f(x)$ has a maximum at $(1, 2)$.

Worked example 6: finding the new equation

The graph of $y = x^2$ is translated 2 units right and 3 units up. Find the new equation.

Moving 2 right means replacing x with $x - 2$. Moving 3 up means adding 3 outside. The new equation is $y = (x-2)^2 + 3$. The turning point moves from $(0, 0)$ to $(2, 3)$. Expanded, $y = x^2 - 4x + 7$.

Standard graphs to sketch

Key fact (Higher): standard graphs

- Cubic $y = x^3$: passes through the origin, rises to the right, falls to the left.
- Reciprocal $y = \dfrac{1}{x}$: two branches, asymptotes along both axes. $y = \dfrac{1}{x} + 2$ has a horizontal asymptote at $y = 2$, and $y = \dfrac{1}{x+1}$ a vertical asymptote at $x = -1$.
- Exponential $y = a^x$ ($a > 1$): passes through $(0, 1)$, always above the x-axis, grows quickly.
- Sine and cosine: wave between -1 and 1 repeating every $360°$. $\sin x$ starts at 0, $\cos x$ starts at 1. The maximum of $y = \sin(x - 30°)$ is at $x = 120°$.

Worked example 7: an exponential graph

A graph $y = ab^x$ passes through $(0, 5)$ and $(1, 15)$. Find the equation and y when $x = 2$.

At $x = 0$, $a = 5$. At $x = 1$, $5b = 15$, so $b = 3$. The equation is $y = 5 \times 3^x$, and when $x = 2$ we get $y = 45$.

Note: exam technique

- Show the substitution line, for example $fg(x) = f(x-3) = \ldots$, so you earn method marks even if you slip later.
- When solving $fg(x) = \text{number}$, expand first and then collect terms to one side before factorising.
- For a "describe fully" transformation, name the type and give the vector or the mirror line.
- When sketching, mark axis crossings, turning points and asymptotes first, move them using the rules and join with a smooth curve.

Watch out: common mistakes

- Doing f first in $fg(x)$. It is g first.
- Moving $f(x+3)$ to the right.
- Thinking $f^{-1}(x)$ means $\dfrac{1}{f(x)}$.
- Writing $f(-x)$ as $-f(x)$: they are different reflections.

Worked example 8: Composite function with a quadratic

$f(x) = x^2 - 1$ and $g(x) = 2x + 3$. Find $fg(x)$ and $gf(x)$.

$fg(x) = f(2x+3) = (2x+3)^2 - 1 = 4x^2 + 12x + 9 - 1 = 4x^2 + 12x + 8$.
$gf(x) = g(x^2-1) = 2(x^2-1) + 3 = 2x^2 + 1$. The two answers are different, so function composition is not commutative.

Worked example 9: Inverse of a function with a bracket

Find the inverse of $f(x) = 2(x-3) + 1$.

Write $y = 2(x-3) + 1$. Subtract 1: $y - 1 = 2(x-3)$. Divide by 2: $\dfrac{y-1}{2} = x - 3$. Add 3: $x = \dfrac{y-1}{2} + 3 = \dfrac{y+5}{2}$. So $f^{-1}(x) = \dfrac{x+5}{2}$. Check with $f(5) = 2 \times 2 + 1 = 5$ and $f^{-1}(5) = 5$.

Worked example 10: The graph of an inverse

The graph of $y = f(x)$ passes through $(2, 7)$. State a point on the graph of $y = f^{-1}(x)$ and describe how the two graphs are related.

Swapping the coordinates gives $(7, 2)$ on $y = f^{-1}(x)$. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$. This is because the inverse reverses the roles of input and output.

Worked example 11: Translating a graph in two directions

The graph of $y = x^2$ is translated by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Write the equation of the new graph.

A translation of 3 to the right replaces x with $x - 3$ and a translation of 2 down subtracts 2: $y = (x - 3)^2 - 2$. The turning point moves from $(0, 0)$ to $(3, -2)$. Notice that the horizontal movement of $+3$ appears as -3 inside the bracket.

Worked example 12: Reflecting in the x-axis and y-axis

The graph of $y = x^3 + x^2$ is reflected in the y-axis. Write the new equation.

Replace x with $-x$: $y = (-x)^3 + (-x)^2 = -x^3 + x^2$. A reflection in the x-axis would multiply the whole function by -1: $y = -x^3 - x^2$. Remember: $f(-x)$ reflects in the y-axis and $-f(x)$ reflects in the x-axis.

Worked example 13: Solving a composite function equation

$f(x) = x + 2$ and $g(x) = x^2$. Solve $gf(x) = 25$.

$gf(x) = (x + 2)^2 = 25$, so $x + 2 = 5$ or $x + 2 = -5$. Therefore $x = 3$ or $x = -7$. Check both: $(3 + 2)^2 = 25$ ✓ and $(-7 + 2)^2 = 25$ ✓. A square equals a positive number for two values.

Worked example 14: Range of a transformed function

$f(x) = x^2$ has range $f(x) \geq 0$. Find the range of $g(x) = 2f(x) - 3$.

The smallest value of $f(x)$ is 0, so the smallest value of $g(x)$ is $2 \times 0 - 3 = -3$. As x gets large, g increases without limit. The range is $g(x) \geq -3$. Stretching vertically by 2 and translating down by 3 moves the minimum value from 0 to -3.

Method: finding the inverse of a function

1. Write the function as $y = \ldots$ using the letter y for the output.
2. Rearrange the equation to make x the subject. Undo the operations in the reverse order: if the function adds 3 and then multiplies by 2, the inverse divides by 2 and then subtracts 3.
3. If x appears twice, collect the x terms on one side and factorise before dividing.
4. Replace y with x and write the answer as $f^{-1}(x) = \ldots$ Check with a number: choose an input, apply f, then apply f^{-1}, and make sure you get back to the start.

Note: how exam questions use functions

In an exam, function questions often use a real situation, such as a taxi fare, a currency exchange or a temperature scale. The function is a rule that turns one

quantity into another, and the inverse works backwards. A composite function describes two steps done one after the other, such as a discount and then a tax. When the question says "show that", write every line of algebra and finish with the expression you were asked to reach. When it says "hence", you must use your previous answer. Keep your working tidy, because a method mark is available even if the arithmetic is wrong.

Note: describing a transformation in words

When you are asked to *describe* a transformation of a graph, name the type of transformation, give its size and give its direction. For example, say "a translation of 2 units to the right" or "a stretch parallel to the y-axis with scale factor 3", never just "it moves". If two transformations are needed, describe them in the order they happen, and remember that a reflection needs the mirror line to be named.

Note: reading transformations from the equation

Changes *inside* the brackets, such as $f(x+3)$ or $f(2x)$, affect the x-coordinates and behave in the opposite way to what you expect: $f(x+3)$ moves the graph 3 **left** and $f(2x)$ squashes it horizontally by a factor of $\frac{1}{2}$. Changes *outside*, such as $f(x)+3$ or $2f(x)$, affect the y-coordinates and behave as you expect. Use a key point such as a turning point to check your answer.

Watch out: common mistakes with functions

- $fg(x)$ means "do g first". Write $f(g(x))$ to avoid confusion.
- $f^{-1}(x)$ is the **inverse** function, not $\dfrac{1}{f(x)}$.
- When solving $f(x)=k$ for a quadratic, remember both the positive and negative square root.
- For domain and range, check for division by zero and negative square roots.

Did you know?

The word *function* was first used in its modern mathematical sense by the German mathematician Gottfried Leibniz in 1673. The notation $f(x)$ that you use today was introduced by the Swiss mathematician Leonhard Euler in 1734. Computers use composite functions all the time: a photo filter applies one function after another to every pixel, so changing the order of filters can give a very different picture.

Otto's tip

For any function question, put a number in first to check your rule. If $f(2)=7$, then $f^{-1}(7)$ must be 2. If it is not, go back and check your algebra.