Answers - Functions and graph transformations

Year: Year 11

Answer to Question 1

a. $f(4) = 12 - 2 =$ **10**; $f(-1) = -3 - 2 = -5$; $f(0.5) = 1.5 - 2 = -0.5$. Replace x by each value: multiply by 3, then subtract 2.
b. $3x - 2 = 19$, so $3x = 21$ and $x = 7$. Set the expression equal to 19, add 2 to both sides, then divide by 3. Check: $f(7) = 21 - 2 = 19$.
c. $g(3) = 9 + 1 =$ **10**; $g(-2) = 4 + 1 =$ **5**. Square first, then add 1; a negative number squared is positive.
d. $x^2 + 1 = 26$, so $x^2 = 25$ and $x = 5$ **or** $x = -5$. Subtract 1 and take the square root, remembering both signs. Check: $25 + 1 = 26$.

Answer to Question 2

a. $fg(x) = f(x^2) = 2x^2 + 3$. $gf(x) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$.
b. $fg(2) = 2 \times 4 + 3 =$ **11**. $gf(2) = 7^2 =$ **49**. Substitute $x = 2$ into the simplified expressions from part (a).
c. $2x^2 + 3 = 35$, so $x^2 = 16$ and $x = 4$ **or** $x = -4$. Subtract 3 from both sides and divide by 2, then take the square root, remembering $\pm$.
d. The order matters: fg means "do g first, then f", which is a different order of operations from gf.

Answer to Question 3

a. Let $y = 4x - 7$. Then $y + 7 = 4x$ and $x = \frac{y + 7}{4}$. So $f^{-1}(x) = \frac{x + 7}{4}$. Undo the operations in reverse order: add 7, then divide by 4. Write the answer using x.
b. $y = \frac{x + 2}{5}$ gives $5y = x + 2$, so $g^{-1}(x) = 5x - 2$. Multiply both sides by 5, then subtract 2.
c. $y = \frac{x}{3} + 1$ gives $y - 1 = \frac{x}{3}$, so $h^{-1}(x) = 3(x - 1) = 3x - 3$. Subtract 1, then multiply by 3.
d. $f^{-1}(9) = \frac{16}{4} =$ **4**. Check: $f(4) = 16 - 7 = 9$. ✓ A function followed by its inverse always returns the starting value.

Answer to Question 4

a. The smallest value is $f(0) = 0$ and the largest is $f(-3) = 9$. The range is $0 \leq f(x) \leq 9$.
b. $x = 0$ gives division by zero, so $x \neq 0$.
c. You cannot take the square root of a negative number, so $x - 1 \geq 0$: domain $x \geq 1$. The output is always at least 0: range $h(x) \geq 0$.
d. When $x = 0$, $k = 1$, and k increases, so the range is $k(x) \geq 1$.

Answer to Question 5

a. Up 2: $(3, 6)$.
b. $f(x + 1)$ moves the graph 1 left: $(2, 4)$.
c. Stretch vertically by 2: $(3, 8)$.
d. Horizontal stretch by scale factor $\frac{1}{2}$: $(1.5, 4)$.
e. Reflection in the x-axis: $(3, -4)$.
f. Reflection in the y-axis: $(-3, 4)$.

Answer to Question 6

a. Roots: $(x - 1)(x - 3) = 0$, so it crosses the x-axis at $x = 1$ and $x = 3$. The y-intercept is 3. Completing the square: $(x - 2)^2 - 1$, so the minimum (turning point) is $(2, -1)$. It is a U-shaped parabola.
b. Passes through $(0, 1)$, $(1, 2)$, $(2, 4)$, $(3, 8)$ and $(-1, 0.5)$. It rises steeply on the right and approaches the x-axis ($y = 0$) on the left but never touches it.
c. Two curved branches, one in the top-right quadrant and one in the bottom-left. It gets close to both axes but never meets them (asymptotes).

Answer to Question 7

a. $2.5 + 1.6 \times 5 =$ **£10.50**. Substitute $d = 5$ into the formula: $1.6 \times 5 = 8$.
b. $y = 2.5 + 1.6d$, so $d = \dfrac{y - 2.5}{1.6}$: $f^{-1}(x) = \dfrac{x - 2.5}{1.6}$. It gives the distance for a given fare.
c. $\dfrac{15.70 - 2.50}{1.6} = \dfrac{13.2}{1.6} =$ **8.25 miles**. Substitute $x = 15.70$ into the inverse function.
d. $gf(d) = 1.1(2.5 + 1.6d) = 2.75 + 1.76d$. For $d = 5$: $2.75 + 8.8 =$ **£11.55**. Apply f first, then g, and expand the bracket.

Answer to Question 8

a. $gh(x) = 1.15(x - 3)$: the euros received for x pounds, after the fee. Apply h first (take off the fee), then g (convert to euros).
b. $1.15 \times 197 =$ **226.55 euros**. Substitute $x = 200$: $200 - 3 = 197$, then multiply by 1.15.
c. $y = 1.15(x - 3)$, so $\frac{y}{1.15} = x - 3$ and $(gh)^{-1}(x) = \frac{x}{1.15} + 3$. Divide both sides by 1.15, then add 3.
d. $\frac{345}{1.15} + 3 = 300 + 3 =$ **£303**. So you need £303 to receive 345 euros.

Answer to Question 9

a. $\frac{5}{9} \times 45 =$ **25°C**. Subtract 32 first: $77 - 32 = 45$, then multiply by $\frac{5}{9}$. So 77 °F is 25 °C.
b. $y = \frac{5}{9}(F - 32)$, so $\frac{9y}{5} = F - 32$ and $C^{-1}(x) = \frac{9x}{5} + 32$. Multiply both sides by $\frac{9}{5}$, then add 32. Write the answer using x.
c. $\frac{9 \times 37}{5} + 32 = 66.6 + 32 =$ **98.6°F**. Substitute $x = 37$ into the inverse: $9 \times 37 = 333$ and $333 \div 5 = 66.6$. Normal body temperature, 37 °C, is 98.6 °F.
d. $C(-40) = \frac{5}{9}(-72) = -40$. ✓ $-40 - 32 = -72$ and $\frac{5}{9} \times (-72) = -40$. So -40 °F is the same as -40 °C.

Answer to Question 10

a. $gf(x) = 1.2 \times 0.9x = 1.08x$. Apply f first (the discount), then g (the VAT), and multiply the two multipliers.
b. $1.08 \times 250 =$ **£270**. The laptop costs £270.
c. $fg(x) = 0.9 \times 1.2x = 1.08x = gf(x)$. Multiplication can be done in either order, so it does **not** matter.
d. $y = 1.08x$, so $(gf)^{-1}(x) = \frac{x}{1.08}$. $\frac{162}{1.08} =$ **£150**. Dividing by 1.08 undoes multiplying by 1.08.

Answer to Question 11

a. $h(0) =$ **20**, $h(1) = 20 - 5 =$ **15**, $h(2) = 20 - 20 =$ **0**.
b. The height goes from 20 down to 0: $0 \leq h(t) \leq 20$.
c. $20 - 5t^2 = 10$, so $5t^2 = 10$, $t^2 = 2$ and $t = 1.41$ **s**.

d. At $t = 2$ the ball reaches the ground ($h = 0$), so the formula no longer applies.

Answer to Question 12

a. $f(x + 4)$ moves the graph 4 units left: $(-2, -3)$.
b. Down 5: $(2, -8)$.
c. Stretch vertically by 2: $(2, -6)$.
d. Horizontal stretch scale factor $\frac{1}{2}$: $(1, -3)$.
e. A reflection in the x-axis turns the minimum into a **maximum** at $(2, 3)$.

Answer to Question 13

a. $0.5,\ 1,\ 2,\ 4,\ 8$.
b. The asymptote moves from $y = 0$ up to $y = 3$. At $x = 0$: $1 + 3 =$ **4**.
c. A translation of 1 unit to the **left** (vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$).
d. $2^{x+1} = 2^x \times 2^1 = 2 \times 2^x$, so shifting left by 1 gives the same graph as stretching vertically by scale factor 2.

Answer to Question 14

a. $15 + 8m = 30 + 5m$, so $3m = 15$ and $m = 5$ **months**. Set the two costs equal, then subtract $5m$ and 15 from both sides.
b. $15 + 40 =$ **£55** on both. Substitute $m = 5$ into either function.
c. $y = 15 + 8m$ gives $m = \dfrac{y - 15}{8}$: $a^{-1}(x) = \dfrac{x - 15}{8}$. $\dfrac{111 - 15}{8} =$ **12 months**. Subtract 15 and divide by 8.
d. $a(12) = 15 + 96 = 111$ and $b(12) = 30 + 60 = 90$. **Plan B** is cheaper by £21.

Answer to Question 15

a. $fg(x) = 3(x^2 + 2) - 1 = 3x^2 + 5$.
$gf(x) = (3x - 1)^2 + 2 = 9x^2 - 6x + 1 + 2 = 9x^2 - 6x + 3$. For fg, put $g(x)$ into f; for gf, put $f(x)$ into g.
b. $3x^2 + 5 = 53$, so $3x^2 = 48$ and $x^2 = 16$: $x = 4$ **or** $x = -4$. Use $fg(x) = 3x^2 + 5$ from part (a).
c. $3x^2 + 5 = 9x^2 - 6x + 3$, so $0 = 6x^2 - 6x - 2$, which divides by 2 to give $3x^2 - 3x - 1 = 0$. Set the two expressions equal and collect all the terms on one side.
d. $x = \dfrac{3 \pm \sqrt{9 + 12}}{6} = \dfrac{3 \pm \sqrt{21}}{6}$, so $x = 1.26$ **or** $x = -0.26$. Use the quadratic

formula with $a = 3$, $b = -3$, $c = -1$.

Answer to Question 16

a. $f(4) = \dfrac{9}{1} = \mathbf{9}$. Substitute $x = 4$: the top is $2 \times 4 + 1 = 9$ and the bottom is $4 - 3 = 1$.

b. Let $y = \dfrac{2x+1}{x-3}$. Then $y(x-3) = 2x + 1$, so $xy - 3y = 2x + 1$. Collect x: $xy - 2x = 1 + 3y$, so $x(y-2) = 1 + 3y$ and $x = \dfrac{3y+1}{y-2}$. Swap the letters: $f^{-1}(x) = \dfrac{3x+1}{x-2}$.

c. $f^{-1}(9) = \dfrac{28}{7} = \mathbf{4}$. ✓ Substitute $x = 9$ into the inverse. It gives back 4, which matches part (a).

d. $x = 2$ would give division by zero, so $x \neq 2$.

Answer to Question 17

a. $g(x) = (x+2)^2 - 5$, with turning point $(-2, -5)$. Replace x by $x + 2$ in x^2, then subtract 5.

b. A translation by the vector $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$. $+2$ inside the bracket moves the graph 2 left, and -5 outside moves it 5 down.

c. $(x+2)^2 = 5$, so $x + 2 = \pm\sqrt{5}$ and $x = -2 \pm 2.236$: $x = 0.24$ **and** $x = -4.24$.

d. The lowest value is -5: $g(x) \geq -5$. A square is never negative, so the smallest value of $g(x)$ is -5.

Answer to Question 18

a. $f(g(x)) = g(x) + 5 = 3x + 2$, so $g(x) = 3x - 3$.

b. $gf(x) = g(x+5) = 3(x+5) - 3 = 3x + 12$. Replace x by $x + 5$ in g.

c. $y = 3x - 3$, so $x = \dfrac{y+3}{3}$: $g^{-1}(x) = \dfrac{x+3}{3}$. Add 3 and divide by 3.

d. $3x - 3 = \dfrac{x+3}{3}$, so $9x - 9 = x + 3$ and $8x = 12$: $x = 1.5$. Check: $g(1.5) = 1.5$ and $g^{-1}(1.5) = 1.5$. ✓

Answer to Question 19

a. $f(x) + 2$: $(0, 2), (2, 6), (4, 2)$. $f(x - 1)$: $(1, 0), (3, 4), (5, 0)$. $-f(x)$:

$(0,0), (2,-4), (4,0)$. $f(2x)$: $(0,0), (1,4), (2,0)$. $2f(x)$: $(0,0), (2,8), (4,0)$.

b. Maximum height 4: the translation $f(x-1)$ and the squash $f(2x)$ both keep the height at 4.

c. $y = f(2x)$, whose points are $(0,0)$, $(1,4)$ and $(2,0)$.

Answer to Question 20

a. $ff(1) = f(f(1)) = f(5) = 17$. So $a + b = 5$ and $5a + b = 17$.

b. Subtract: $4a = 12$, so $a = 3$ and $b = 2$. Check: $f(x) = 3x + 2$, $f(1) = 5$, $f(5) = 17$. ✓

c. $y = 3x + 2$, so $f^{-1}(x) = \dfrac{x-2}{3}$. Subtract 2 and divide by 3.

d. $3x + 2 = \dfrac{x-2}{3}$, so $9x + 6 = x - 2$ and $8x = -8$: $x = -1$. Check: $f(-1) = -1$ and $f^{-1}(-1) = -1$. ✓