Lesson - Equation of a circle and tangents

Year: Year 11

This whole topic is Higher tier. It joins coordinate geometry, Pythagoras and perpendicular lines. At GCSE the circle always has its centre at the origin $(0, 0)$.

Starter activity

Draw a circle of radius 5 on squared paper with its centre at the origin. Mark every point with whole-number coordinates that lies on the circle, such as $(3, 4)$. Check that $x^2 + y^2 = 25$ for each one. Now draw the radius to $(3, 4)$ and use a set square to draw the line at right angles to it at that point. What do you notice about this line and the circle?

The equation of a circle

Theorem (Higher): equation of a circle

Take any point (x, y) on a circle of radius r with centre $(0, 0)$. The horizontal distance from the centre is x and the vertical distance is y. By Pythagoras, $x^2 + y^2 = r^2$:

$$x^2 + y^2 = r^2$$

The number on the right is the radius *squared*. The circle $x^2 + y^2 = 25$ has radius 5. The circle $x^2 + y^2 = 10$ has radius $\sqrt{10}$.

Definition (Higher): points on, inside and outside

A point can be **on**, **inside** or **outside** a circle. Substitute the coordinates into $x^2 + y^2$ and compare with r^2.

- If $x^2 + y^2 = r^2$ the point is **on** the circle.
- If $x^2 + y^2 < r^2$ the point is **inside**.
- If $x^2 + y^2 > r^2$ the point is **outside**.

Worked example 1: finding the equation

Find the equation of the circle, centre the origin, passing through $(-5, 12)$.

$r^2 = (-5)^2 + 12^2 = 25 + 144 = 169$. The equation is $x^2 + y^2 = 169$, so the radius is 13.

Tangents

Definition: tangent

A **tangent** is a straight line that touches the circle at exactly one point.

Rule: tangent and radius

A **tangent is perpendicular to the radius** at the point of contact. The gradients multiply to give -1: if the radius has gradient m, the tangent has gradient $-\frac{1}{m}$.

Method: finding the equation of a tangent

1. Check that the point lies on the circle.
2. Find the gradient of the radius from $(0, 0)$ to the point: $\frac{y}{x}$.
3. Find the perpendicular gradient: negative reciprocal.
4. Use $y - y_1 = m(x - x_1)$ and tidy up.

Worked example 2: tangent at $(3, 4)$

Find the tangent to $x^2 + y^2 = 25$ at $(3, 4)$.

Check: $9 + 16 = 25$. Radius gradient $= \frac{4}{3}$. Tangent gradient $= -\frac{3}{4}$.

$y - 4 = -\frac{3}{4}(x - 3) \Rightarrow 4y - 16 = -3x + 9 \Rightarrow 3x + 4y = 25$.

Worked example 3: tangent at $(-6, 8)$

Find the tangent to $x^2 + y^2 = 100$ at $(-6, 8)$.

Radius gradient $= \frac{8}{-6} = -\frac{4}{3}$. Tangent gradient $= \frac{3}{4}$.

$y - 8 = \frac{3}{4}(x + 6) \Rightarrow 4y - 32 = 3x + 18 \Rightarrow 3x - 4y = -50$, or $y = \frac{3}{4}x + 12.5$.

Worked example 4: tangent at $(4, 2)$

Find the tangent to $x^2 + y^2 = 20$ at $(4, 2)$ and where it crosses the x-axis.

Check: $16 + 4 = 20$, so the point is on the circle. The radius gradient is $\frac{2}{4} = \frac{1}{2}$, so the tangent gradient is -2. Then $y - 2 = -2(x - 4)$, which gives $y = -2x + 10$. It crosses the x-axis where $0 = -2x + 10$, so at $x = 5$.

Watch out: points on an axis

If the point is on an axis, the radius is horizontal or vertical, so the tangent is vertical or horizontal. For $x^2 + y^2 = 16$, the tangent at $(0, 4)$ is the line $y = 4$ and the tangent

at $(-4, 0)$ is the line $x = -4$. Do not use $-\frac{1}{m}$ here because a vertical line has no gradient.

Key fact (Higher): why the method works

The radius to the point (a, b) has gradient $\frac{b}{a}$. The tangent meets it at 90°, so it has gradient $-\frac{a}{b}$. Putting this into $y - b = -\frac{a}{b}(x - a)$ and tidying gives $ax + by = a^2 + b^2 = r^2$. For $(3, 4)$ on $x^2 + y^2 = 25$ this gives $3x + 4y = 25$. Use this quick form to *check* your answer, but show the gradient method in the exam to earn the marks.

Where a line meets a circle

Rule (Higher): line and circle

To find where a line crosses a circle, substitute the line into the circle equation and solve the quadratic.

- Two solutions: the line cuts the circle at two points.
- One (repeated) solution: the line is a tangent.
- No real solutions: the line misses the circle.

Worked example 5: a line meeting a circle

Find where $y = x + 1$ meets $x^2 + y^2 = 25$.

$x^2 + (x + 1)^2 = 25 \Rightarrow 2x^2 + 2x - 24 = 0 \Rightarrow x^2 + x - 12 = 0 \Rightarrow (x + 4)(x - 3) = 0$. If $x = 3$, $y = 4$. If $x = -4$, $y = -3$. The points are $(3, 4)$ and $(-4, -3)$.

Reasoning questions

Worked example 6: showing a tangent and tangent length

Show that $y = 5$ is a tangent to $x^2 + y^2 = 25$. Find the tangent length from $P = (5, 0)$ to the circle $x^2 + y^2 = 20$.

Substitute $y = 5$: $x^2 + 25 = 25$, so $x^2 = 0$ and there is only one solution, $x = 0$. One point of contact means a tangent, touching at $(0, 5)$. State the conclusion in words. For the tangent length, triangle OTP has a right angle at T, so $PT^2 = OP^2 - r^2 = 25 - 20 = 5$ and $PT = \sqrt{5}$.

Watch out: common mistakes

- Writing the radius instead of the radius squared: $x^2 + y^2 = 5$ is not a circle of radius 5.
- Using the radius gradient as the tangent gradient. Take the negative reciprocal.
- Forgetting to flip *and* change the sign: the perpendicular of $\frac{4}{3}$ is $-\frac{3}{4}$, not $\frac{3}{4}$ and not $-\frac{4}{3}$.
- Not checking that the point is actually on the circle.

Worked example 7: Finding the centre by completing the square

Find the centre and radius of the circle $x^2 + y^2 - 4x + 6y - 12 = 0$.

Group the terms: $(x^2 - 4x) + (y^2 + 6y) = 12$. Complete each square: $(x - 2)^2 - 4 + (y + 3)^2 - 9 = 12$. Rearrange: $(x - 2)^2 + (y + 3)^2 = 25$. The centre is $(2, -3)$ and the radius is 5. Always add the numbers you subtracted to the right-hand side.

Worked example 8: Perpendicular gradients

A radius has gradient $-\frac{2}{5}$. What is the gradient of the tangent at the end of the radius?

The tangent is perpendicular to the radius, so its gradient is the negative reciprocal: flip the fraction and change the sign. $-\frac{2}{5}$ becomes $\frac{5}{2}$. Check: $-\frac{2}{5} \times \frac{5}{2} = -1$. If the radius is horizontal, the tangent is vertical, and vice versa.

Worked example 9: Finding where a line is a tangent

Show that the line $y = 3$ is a tangent to the circle $(x - 1)^2 + (y - 1)^2 = 4$ and find the point of contact.

Substitute $y = 3$: $(x - 1)^2 + 4 = 4$, so $(x - 1)^2 = 0$ and $x = 1$. There is exactly one solution, so the line touches the circle at one point: $(1, 3)$. This is the point directly above the centre $(1, 1)$ at a distance 2, which is the radius.

Worked example 10: A tangent through a point on the circle with a negative gradient

Find the tangent at $(4, -3)$ to the circle $x^2 + y^2 = 25$.

Gradient of the radius $= \frac{-3}{4}$. Tangent gradient $= \frac{4}{3}$. $y + 3 = \frac{4}{3}(x - 4)$, so $3y + 9 = 4x - 16$ and $4x - 3y = 25$. Check $(4, -3)$: $16 + 9 = 25$ ✓. The coefficients are the coordinates of the point, which gives the shortcut $4x - 3y = 25$ for the circle $x^2 + y^2 = 25$ at $(4, -3)$.

Worked example 11: The length of a tangent

A tangent is drawn from the point $(0, 13)$ to the circle $x^2 + y^2 = 25$. Find its length.

The distance from the centre to the external point is 13. The radius to the point of contact is perpendicular to the tangent, forming a right-angled triangle with hypotenuse 13 and one side 5. The tangent length is $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$.

Worked example 12: Intersection of two curves

Find where the circle $x^2 + y^2 = 20$ meets the line $y = 2x$.

Substitute: $x^2 + 4x^2 = 20$, so $5x^2 = 20$ and $x^2 = 4$, giving $x = \pm 2$. The points are $(2, 4)$ and $(-2, -4)$. They are opposite ends of a diameter because the line passes through the centre.

Worked example 13: Equation of a circle from three pieces of information

A circle has centre $(2, -1)$ and passes through $(5, 3)$. Find its equation and decide whether $(2, 4)$ is inside the circle.

$r^2 = (5 - 2)^2 + (3 + 1)^2 = 9 + 16 = 25$, so the equation is $(x - 2)^2 + (y + 1)^2 = 25$. For $(2, 4)$: $(0)^2 + (5)^2 = 25$, which equals r^2, so the point is **on** the circle, not inside.

Worked example 14: A full tangent question with a diagram

A circle has centre $O(0, 0)$ and radius $\sqrt{40}$. The point $A(6, 2)$ is on the circle. Find the equation of the tangent at A and the point where it crosses the y-axis.

Check: $6^2 + 2^2 = 40$ ✓. The gradient of OA is $\frac{2}{6} = \frac{1}{3}$, so the tangent gradient is -3. Using $y - 2 = -3(x - 6)$ gives $y = -3x + 20$. Where $x = 0$, $y = 20$, so the tangent crosses the y-axis at $(0, 20)$. Notice the shortcut for a circle centred at the origin: the tangent at (a, b) is $ax + by = r^2$, here $6x + 2y = 40$, which simplifies to $3x + y = 20$. Both forms agree.

Note: three standard question types

Questions on this topic nearly always ask one of three things: (1) write or read the equation of a circle, (2) find the equation of a tangent at a point, using perpendicular gradients, or (3) find where a line meets a circle by substituting and solving a quadratic. For type 3, the discriminant tells you how many intersections there are: $b^2 - 4ac > 0$ means two points, $= 0$ means a tangent, and < 0 means no intersection.

Watch out: sign and radius errors

- The equation $(x-3)^2+(y+2)^2=25$ has centre $(3,-2)$, not $(-3,2)$. The signs in the brackets are the opposite of the centre.
- The right-hand side is r^2, so the radius of $x^2+y^2=49$ is 7, not 49.
- The tangent gradient is the **negative reciprocal**, not just the negative.
- When the radius is horizontal or vertical, use $x=\ldots$ or $y=\ldots$ for the tangent; do not divide by zero.

Did you know?

The equation of a circle is a direct use of Pythagoras\' theorem: a point (x,y) is at distance r from the centre exactly when $x^2+y^2=r^2$. The French mathematician and philosopher René Descartes, who lived in the 17th century, was the first to describe shapes with equations like this by putting them on a grid. That is why the grid you use in school is called a Cartesian grid.

Otto's tip

Draw a quick sketch with the circle, the point and a rough tangent. A tangent at a point in the top right has a negative gradient, so if your answer is positive, something has gone wrong. Check the equation by substituting the point back in.