Answers - Equation of a circle and tangents

Year: Year 11

Answer to Question 1

a. $x^2 + y^2 = 25$. With centre $(0, 0)$ the brackets become x^2 and y^2, and the right-hand side is $r^2 = 5^2 = 25$.
b. $(x - 2)^2 + (y - 3)^2 = 16$. Substitute $a = 2$, $b = 3$ and $r^2 = 4^2 = 16$ into $(x - a)^2 + (y - b)^2 = r^2$.
c. $(x + 1)^2 + (y - 4)^2 = 9$ (subtracting -1 gives $+1$). Substitute $a = -1$, $b = 4$ and $r^2 = 9$.
d. $x^2 + y^2 = 20$. Squaring $\sqrt{20}$ gives 20, so the right-hand side is 20.
e. $r^2 = (6 - 3)^2 + (-1 + 5)^2 = 9 + 16 = 25$, so $(x - 3)^2 + (y + 5)^2 = 25$. The radius is the distance from the centre to the given point, so we find r^2 with Pythagoras.

Answer to Question 2

a. Centre $(0, 0)$, radius **7**. Compare with $x^2 + y^2 = r^2$: $r^2 = 49$, so $r = 7$.
b. Centre $(3, -2)$, radius **5**. The signs inside the brackets are the opposite of the coordinates of the centre, and $r = \sqrt{25}$.
c. Centre $(-4, 0)$, radius $\sqrt{12} = 2\sqrt{3}$. $x + 4 = x - (-4)$ and $y^2 = (y - 0)^2$. Simplify $\sqrt{12} = \sqrt{4 \times 3}$.
d. Centre $(1, 1)$, radius $\sqrt{18} = 3\sqrt{2}$. Simplify $\sqrt{18} = \sqrt{9 \times 2}$.

Answer to Question 3

a. $(3, 4)$: $9 + 16 = 25$. $(5, 1)$: $25 + 1 = 26$. $(2, -3)$: $4 + 9 = 13$. $(-4, -3)$: $16 + 9 = 25$. $(4, 4)$: $16 + 16 = 32$.
b. **On:** $(3, 4)$ and $(-4, -3)$. **Outside:** $(5, 1)$ and $(4, 4)$ (greater than 25). **Inside:** $(2, -3)$ (less than 25).
c. $(2, -3)$ has the smallest value, 13, so it is closest to the centre. So $(2, -3)$ is the closest point, at a distance $\sqrt{13} \approx 3.61$ from the centre.

Answer to Question 4

a. Gradient of $OA = \dfrac{4 - 0}{3 - 0} = \dfrac{4}{3}$. Use gradient $= \dfrac{\text{change in } y}{\text{change in } x}$ from $O(0, 0)$ to $A(3, 4)$

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b. Perpendicular gradient $= -1 \div \frac{4}{3} = -\frac{3}{4}$. Perpendicular gradients multiply to -1, so the tangent gradient is the negative reciprocal of $\frac{4}{3}$.
c. $y - 4 = -\frac{3}{4}(x - 3)$. Multiply by 4: $4y - 16 = -3x + 9$, so $3x + 4y = 25$. Check: $3 \times 3 + 4 \times 4 = 25$. Use $y - y_1 = m(x - x_1)$ with the point A, then clear the fraction.

Answer to Question 5

a. $(-6)^2 + 8^2 = 36 + 64 = 100$. The point satisfies the equation, so P is on the circle.
b. Gradient of $OP = \frac{8}{-6} = -\frac{4}{3}$. The tangent gradient is the negative reciprocal: $\frac{3}{4}$. The tangent is perpendicular to the radius OP.
c. $y - 8 = \frac{3}{4}(x + 6)$, so $y = \frac{3}{4}x + 4.5 + 8 = y = \frac{3}{4}x + 12.5$. Check: when $x = -6$, $y = -4.5 + 12.5 = 8$. Use $y - y_1 = m(x - x_1)$ with the point $P(-6, 8)$, then expand and add 8.

Answer to Question 6

a. $(5, 0)$, $(-5, 0)$, $(0, 5)$, $(0, -5)$. The circle has radius 5 and centre O, so it meets each axis 5 units from the origin.
b. The radius to $(5, 0)$ is horizontal, so the tangent is vertical: $x = 5$. A tangent is perpendicular to the radius at the point of contact, and every point on a vertical line has the same x-coordinate.
c. The radius to $(0, -5)$ is vertical, so the tangent is horizontal: $y = -5$. Every point on a horizontal line has the same y-coordinate.
d. The tangents $x = \pm 5$ and $y = \pm 5$ form a **square** of side 10, area **100**. Each side of the square runs from -5 to 5, so the side is 10 and the area is $10^2 = 100$ square units.

Answer to Question 7

a. $x^2 + (y - 10)^2 = 100$. The centre is $(0, 10)$ and the radius is 10, so $r^2 = 100$.
b. $6^2 + (y - 10)^2 = 100$, so $(y - 10)^2 = 64$ and $y - 10 = \pm 8$. The heights are $y = 18$ **m** and $y = 2$ **m**. Substitute $x = 6$, subtract 36 from both sides and take the square root, remembering $\pm$. The pod is either 18 m or 2 m above the ground.
c. $6^2 + (2 - 10)^2 = 36 + 64 = 100$, so $(6, 2)$ is on the circle. The point satisfies the equation, so the pod is on the wheel.
d. The lowest point is $(0, 0)$ ($y - 10 = -10$): it **touches the ground** (0 m). The lowest point is directly below the centre, one radius down: $10 - 10 = 0$. So the

wheel just touches the ground.

Answer to Question 8

a. $x^2 + y^2 = 50^2 = x^2 + y^2 = 2500$. The radar range is a circle with centre O and radius 50 km.
b. $30^2 + 40^2 = 900 + 1600 = 2500$, so it is **on the boundary**. Compare $x^2 + y^2$ with $r^2 = 2500$: equal means on the circle.
c. $35^2 + 40^2 = 2825 > 2500$, so it is **outside**: not detected. Its distance is $\sqrt{2825} = 53.15$, which is $53.15 - 50 =$ **3.15 km** beyond the edge. A value bigger than 2500 means the ship is further than 50 km from O.
d. $20^2 + 30^2 = 1300 < 2500$, so it is **inside** and detected. $\sqrt{1300} \approx 36.1$ km, which is less than 50 km.

Answer to Question 9

a. $(x-3)^2 + (y+2)^2 = 36$. The centre is $(3, -2)$, so the brackets are $(x-3)$ and $(y+2)$, and $r^2 = 6^2 = 36$.
b. $(3-3)^2 + (4+2)^2 = 0 + 36 = 36$. ✓ The point satisfies the equation, so it is on the boundary.
c. Put $y = -2$: $(x-3)^2 = 36$, so $x = 9$ or $x = -3$. The point on the right is $(9, -2)$.
d. $(6, 2)$: $(3)^2 + (4)^2 = 25 < 36$, so it is **inside** (watered). $(8, 3)$: $(5)^2 + (5)^2 = 50 > 36$, so it is **outside** (not watered).

Answer to Question 10

a. $24^2 + 32^2 = 576 + 1024 = 1600 = 40^2$. The point satisfies $x^2 + y^2 = 40^2$, so A is on the track.
b. Gradient of $OA = \dfrac{32}{24} = \dfrac{4}{3}$; tangent gradient $-\dfrac{3}{4}$. $y - 32 = -\dfrac{3}{4}(x - 24)$, so $4y - 128 = -3x + 72$ and $3x + 4y = 200$. The tangent is perpendicular to the radius OA. Use $y - y_1 = m(x - x_1)$, then multiply by 4. Check: $72 + 128 = 200$.
c. $y = 0$: $x = \dfrac{200}{3} = 66.7$. $x = 0$: $y = 50$. So $(66.7, 0)$ and $(0, 50)$. Put $y = 0$ to find where the tangent meets the x-axis, and $x = 0$ for the y-axis.
d. From $(24, 32)$ to $(0, 50)$: $\sqrt{24^2 + 18^2} = \sqrt{900} =$ **30 m**. Use Pythagoras with the changes 24 across and $50 - 32 = 18$ up. The runner travels 30 m.

Answer to Question 11

a. Substitute $y = x + 1$: $x^2 + (x+1)^2 = 25$, so $2x^2 + 2x + 1 = 25$, $2x^2 + 2x - 24 = 0$, so $x^2 + x - 12 = 0$. Replace y by $x + 1$, expand $(x+1)^2 = x^2 + 2x + 1$, subtract 25 and divide by 2.
b. $(x+4)(x-3) = 0$, so $x = 3$ or $x = -4$. The points are $(3, 4)$ and $(-4, -3)$. We need two numbers with product -12 and sum 1: 4 and -3. Find each y from $y = x + 1$.
c. Distance $= \sqrt{(3-(-4))^2 + (4-(-3))^2} = \sqrt{49+49} = \sqrt{98} =$ **9.90 m**. The changes are 7 across and 7 up, so we use Pythagoras. About 9.90 m of the path is inside the pond.

Answer to Question 12

a. $x^2 + (x+4)^2 = 8$, so $2x^2 + 8x + 16 = 8$, so $2x^2 + 8x + 8 = 0$, so $x^2 + 4x + 4 = 0$.
b. $(x+2)^2 = 0$, so $x = -2$ is a repeated root: one point of contact only. A line meeting a circle at exactly one point is a tangent.
c. $y = -2 + 4 = 2$, so the point of contact is $(-2, 2)$.
d. Gradient of the radius $= \dfrac{2}{-2} = -1$. Gradient of the line $= 1$. $-1 \times 1 = -1$, so they are perpendicular.

Answer to Question 13

a. The radius OA is perpendicular to the tangent TA. A tangent is always perpendicular to the radius at the point of contact.
b. $OT = 13$ and $OA = 5$. $TA^2 = 13^2 - 5^2 = 144$, so $TA =$ **12 m**. $OT = 13$ is the hypotenuse because it is opposite the right angle.
c. $\sin(ATO) = \dfrac{5}{13}$, so $ATO = 22.6°$. $OA = 5$ is opposite angle ATO and $OT = 13$ is the hypotenuse, so we use sine.
d. Two congruent triangles: $2 \times \frac{1}{2} \times 5 \times 12 =$ **60 m²**. The kite is made of the two right-angled triangles OAT and OBT, each with base 5 and height 12.

Answer to Question 14

a. Angle $AOB = 90°$ because the axes are perpendicular, and the angle in a semicircle is $90°$ (converse), so AB is a diameter.
b. The centre is the midpoint of AB: $(4, 3)$. $AB = \sqrt{64+36} = 10$, so the radius is **5 m**.
c. $(x-4)^2 + (y-3)^2 = 25$.
d. $(4-4)^2 + (8-3)^2 = 0 + 25 = 25$ ✓. Area $= \pi \times 5^2 =$ **78.5 m²**.

Answer to Question 15

a. $(5-1)^2+(4-2)^2=16+4=20$. The point satisfies the equation, so P is on the circle.

b. Gradient $=\dfrac{4-2}{5-1}=\dfrac{1}{2}$. Use the centre $C(1,2)$ and the point $P(5,4)$.

c. Tangent gradient $=-2$. $y-4=-2(x-5)$, so $y=-2x+14$. The tangent is perpendicular to the radius, so its gradient is the negative reciprocal of $\frac{1}{2}$. Check: $-2\times5+14=4$.

d. $y=0$: $x=7$, so $A(7,0)$. $x=0$: $y=14$, so $B(0,14)$. Area $=\frac{1}{2}\times7\times14=$ **49 square units**.

Answer to Question 16

a. Circle: $36+4=40$ and $4+36=40$. Line: $-6+8=2$ and $-2+8=6$. ✓ A point lies on a graph when its coordinates satisfy the equation.

b. $AB=\sqrt{(6-2)^2+(2-6)^2}=\sqrt{32}=$ **5.66**. Use Pythagoras with changes of 4 across and 4 down.

c. $M=\left(\dfrac{6+2}{2},\dfrac{2+6}{2}\right)=(4,4)$. Gradient $OM=1$ and the gradient of the line is -1; $1\times(-1)=-1$, so $OM\perp AB$. The product of the gradients is -1, so OM is perpendicular to the chord.

d. $OM=\sqrt{16+16}=\sqrt{32}=5.66$. Check: $OA^2=OM^2+AM^2=32+8=40$. ✓ AM is half of AB, so $AM^2=8$. The radius squared, 40, equals $32+8$.

Answer to Question 17

a. Midpoint of $AB=\left(\dfrac{-2+6}{2},\dfrac{3+9}{2}\right)=(2,6)$. The centre of a circle is the midpoint of any diameter.

b. $AB=\sqrt{(6+2)^2+(9-3)^2}=\sqrt{100}=10$, so the radius is **5**. The radius is half of the diameter AB, which we find with Pythagoras.

c. $(x-2)^2+(y-6)^2=25$. Check with A: $16+9=25$. Use $(x-a)^2+(y-b)^2=r^2$ with centre $(2,6)$ and $r^2=25$.

d. $(2-2)^2+(11-6)^2=0+25=25$. ✓ The point satisfies the equation, so it lies on the circle.

Answer to Question 18

a. $PA^2=OP^2-OA^2=100-25=75$, so $PA=\sqrt{75}=$ **8.66**. OA is perpendicular to the tangent PA, so triangle OAP is right-angled at A with hypotenuse

$OP = 10$. Exactly, $PA = 5\sqrt{3}$.

b. $\sin(OPA) = \dfrac{5}{10} = 0.5$, so $OPA = 30°$. $OA = 5$ is opposite angle OPA and $OP = 10$ is the hypotenuse, so we use sine.

c. By symmetry the tangents make equal angles with OP, so $APB = 2 \times 30 = 60°$. OP is a line of symmetry of the figure.

d. Two right-angled triangles each of area $\frac{1}{2} \times 5 \times 8.66 = 21.65$, so the area is **43.30 square units**. The exact area is $25\sqrt{3} \approx 43.30$.

Answer to Question 19

a. $x^2 + (2x + k)^2 = 5$ gives $x^2 + 4x^2 + 4kx + k^2 = 5$, so $5x^2 + 4kx + k^2 - 5 = 0$. Replace y by $2x + k$ and expand $(2x + k)^2 = 4x^2 + 4kx + k^2$.

b. $b^2 - 4ac = (4k)^2 - 4 \times 5 \times (k^2 - 5) = 16k^2 - 20k^2 + 100 = 100 - 4k^2 = 0$, so $k^2 = 25$. Here $a = 5$, $b = 4k$ and $c = k^2 - 5$. A tangent meets the circle at one point, so the discriminant is zero.

c. $k = 5$ **or** $k = -5$. For $k = 5$: $5x^2 + 20x + 20 = 0$, so $x^2 + 4x + 4 = 0$, $x = -2$ and $y = 2(-2) + 5 = 1$: the point of contact is $(-2, 1)$ (check: $4 + 1 = 5$). Take the square root of 25, remembering both signs. Substitute $k = 5$ and divide by 5: $(x + 2)^2 = 0$, so there is one repeated root.

Answer to Question 20

a. $x^2 - 6x = (x - 3)^2 - 9$ and $y^2 - 8y = (y - 4)^2 - 16$. So $(x - 3)^2 + (y - 4)^2 = 25$: centre $(3, 4)$, radius **5**.

b. At $(0, 0)$: $9 + 16 = 25$ ✓. Gradient of the radius $= \dfrac{4}{3}$, tangent gradient $-\dfrac{3}{4}$ through the origin: $y = -\dfrac{3}{4}x$.

c. $y = 0$: $x^2 - 6x = 0$, so $x = 6$: $A(6, 0)$. $x = 0$: $y^2 - 8y = 0$, so $y = 8$: $B(0, 8)$. Angle $AOB = 90°$ (the axes), and an angle of $90°$ at the circumference stands on a diameter. Check: $AB = \sqrt{36 + 64} = 10 = 2 \times 5$. ✓