Lesson - Constructions, loci, bearings, plans and elevations

Year: Year 11

This topic is on both the Foundation and Higher papers. It mixes drawing skills (ruler, compasses and protractor) with reasoning about position and direction. Always leave your construction arcs visible.

Starter activity

Stand at the front of your classroom and face the board. Suppose that is north. Walk to the back-left corner, then to the back-right corner. For each walk, estimate the bearing. How would you check using a back bearing? Now think about how you would draw the same journey on squared paper with a scale of 1 cm to 2 m, and how you would measure the straight-line distance back to the start.

Ruler and compass constructions

Method: perpendicular bisector of a line AB

1. Open the compasses to more than half the length of AB.
2. Put the point on A and draw an arc above and below the line.
3. Keep the same radius, put the point on B and draw two more arcs crossing the first two.
4. Join the two crossing points with a ruler. This line cuts AB in half at 90°.

Every point on the perpendicular bisector is the same distance from A as from B.

Method: bisector of an angle

1. Put the compass point on the corner and draw an arc crossing both arms.
2. From each crossing point draw an arc of the same radius so the two arcs meet inside the angle.
3. Join the corner to where the arcs meet.

Every point on an angle bisector is the same distance from the two arms.

Key fact: perpendiculars

Perpendicular from a point to a line: draw an arc from the point that cuts the line twice, then bisect between the two cuts. **Perpendicular at a point on a line**: mark equal distances either side of the point, then draw a perpendicular bisector of that short piece.

Loci

Definition: locus

A **locus** (plural loci) is the set of all points that obey a rule.

Rule: the four standard loci

- A fixed distance from a point: a **circle**.
- A fixed distance from a line segment: a **racetrack shape** with two straight sides and two semicircular ends.
- Equidistant from two points: the **perpendicular bisector**.
- Equidistant from two lines: the **angle bisector**.

When a region is described by two rules, draw each locus lightly and shade the part where both rules are true. Use a dashed line for a boundary that is not included ("less than 3 cm").

Worked example 1: region from loci

Find the area of points within 2 cm of a line segment of length 5 cm.

The region is a rectangle $5 \times 4 = 20\ \text{cm}^2$ plus two semicircles that make one full circle of radius 2: $\pi \times 2^2 = 12.57$. Total $= 32.57\ \text{cm}^2$.

Bearings

Definition: bearing

A **bearing** is an angle measured **clockwise from north** and always written with **three figures**. North is 000°, east 090°, south 180°, west 270°. A bearing of 45° is written 045°.

Method: bearing questions

1. Put your finger on the starting point.
2. Draw a north line there.
3. Measure clockwise and write three figures.

Rule: back bearings

Add 180° if the bearing is less than 180°; subtract 180° if it is more. If the bearing of B from A is 070°, the bearing of A from B is 250°. North lines are parallel, so co-interior (allied) angles add up to 180°.

Worked example 2: a ship's journey

A ship sails 12 km from P to Q on a bearing of 060°, then 5 km from Q to R on a bearing of 150°. Find PR and the bearing of R from P.

The back bearing from Q to P is $060 + 180 = 240^\circ$. The angle between QP (240°) and QR (150°) is $240 - 150 = 90^\circ$. So triangle PQR is right-angled at Q and $PR = \sqrt{12^2 + 5^2} = 13$ km. The bearing of R from P is $060^\circ + \tan^{-1}\dfrac{5}{12} \approx 082.6^\circ$.

Scale drawings and maps

Rule: scales

A scale of $1 : 50\,000$ means 1 cm on the map is 50 000 cm on the ground, which is 500 m. To find real distance, multiply the map length by the scale. To find map length, divide the real length by the scale. Convert units at the end: 100 000 cm $= 1$ km.

Worked example 3: a map path

On a $1 : 25\,000$ map, a path is 14 cm long. Find its real length.

Real length $= 14 \times 25\,000 = 350\,000$ cm $= 3.5$ km.

Plans and elevations

Definition: plan and elevation

A **plan** is the view from above. A **front elevation** is the view from the front and a **side elevation** is the view from the side. All are drawn flat, showing true lengths with no perspective.

Worked example 4: views of solids

Give the plan and elevations of a cuboid $4\text{ cm} \times 3\text{ cm} \times 2\text{ cm}$ high, a cone and a triangular prism lying on a rectangular face.

The cuboid has a plan 4×3, a front elevation 4×2 and a side elevation 3×2. A cone has a circular plan and a triangular elevation. The prism has a rectangular plan and one rectangular elevation, and its other elevation is a triangle.

Watch out: common mistakes

- Measuring a bearing anticlockwise or from the wrong north line.
- Writing a bearing with fewer than three figures.
- Rubbing out construction arcs.
- Changing the compass radius between the two arcs of a bisector.

- Forgetting to convert cm to km at the end of a scale question.

Worked example 5: Back bearings

The bearing of P from Q is 305°. Find the bearing of Q from P.

305 is greater than 180, so subtract 180: $305 - 180 = 125^\circ$. Check by drawing: from Q the point P is north-west, so from P the point Q is south-east, which is a bearing near 135°. The answer 125° is in the correct quadrant. Always sketch a north line at each point.

Worked example 6: Converting a scale

A model of a ship is built at a scale of $1 : 80$. The model is 45 cm long. Find the length of the real ship in metres.

$45 \times 80 = 3600$ cm. Divide by 100 to change to metres: $3600 \div 100 = 36$ m. Always convert to a convenient unit at the end. For areas the scale factor is squared: 1 cm^2 on the model is $80^2 = 6400$ cm^2 on the ship.

Worked example 7: Constructing a 60° angle

Describe how to construct a 60° angle at point A on a line using only compasses and a ruler.

Place the compass point at A and draw an arc across the line at B. Keep the same radius, put the point at B and draw an arc crossing the first arc at C. Join A to C. Triangle ABC has three equal sides, so it is equilateral and angle $BAC = 60^\circ$. Bisecting it gives 30°.

Worked example 8: A locus of equal distance from two lines

Lines AB and AC meet at A with angle $BAC = 50^\circ$. Describe the locus of points equidistant from the two lines and state the angle it makes with each.

The locus is the **angle bisector** of BAC. To draw it, put the compass point at A and draw an arc across both lines. From the two crossing points draw arcs of the same radius that meet. Draw a line from A through the meeting point. It makes 25° with each line. Every point on it is the same perpendicular distance from the two lines.

Worked example 9: Using the plan and elevations of a solid

The plan of a solid is a circle of diameter 6 cm, the front elevation is a rectangle 6 cm by 10 cm. Name the solid and find its volume.

A circle from above and a rectangle from the front mean a **cylinder** of diameter 6 cm and height 10 cm. Radius $= 3$ cm. Volume $= \pi r^2 h = \pi \times 3^2 \times 10 = 90\pi = 282.7$ cm^3. If the front elevation were a triangle, the solid would be a cone.

Worked example 10: A bearing in a triangle of sides

A walker goes 5 km due east then 12 km due north. Find the bearing of the start from the walker.

The final position is $(5, 12)$. The bearing of the walker from the start is $\tan^{-1}(5/12) = 22.6°$, that is $023°$. The bearing of the start from the walker is the back bearing $22.6 + 180 = 202.6 \approx 203°$. The distance is $\sqrt{5^2 + 12^2} = 13$ km.

Worked example 11: Area on a scale drawing

On a plan with scale $1 : 200$, a rectangular car park measures 4 cm by 3 cm. Find the real area in square metres.

The real sides are $4 \times 200 = 800$ cm $= 8$ m and $3 \times 200 = 600$ cm $= 6$ m. Area $= 8 \times 6 = 48$ m^2. Alternatively the plan area is 12 cm^2 and the scale factor for area is $200^2 = 40\,000$, giving $480\,000$ cm$^2 = 48$ m^2.

Note: marks for constructions

In an exam you must use a ruler and compasses and **leave in your construction arcs**. Do not rub them out. Arcs should cross clearly, with the same compass setting for both arcs from each endpoint. For loci questions, shade the required region neatly and say whether the boundary is included. Check your result by measuring or by calculating one length with Pythagoras.

Watch out: errors with bearings and scales

- Bearings are measured **clockwise from north** and written with three figures: $045°$, not $45°$.
- When you draw the angle at the second point, draw a new north line there (parallel to the first).
- For a scale of $1 : n$ multiply by n to go from the map to real life and divide by n the other way.
- Areas scale by the **square** of the length scale factor.

Did you know?

The word *locus* is Latin for "place" and its plural is *loci*. Mathematicians have used loci since the ancient Greeks, who defined a circle as the locus of points equidistant from a point. Sailors have used bearings for navigation for hundreds of years and a three-figure bearing is still the standard way for pilots and ship captains to give direction, so the numbers you use in this chapter are the same as those used in real air traffic control.

Otto's tip

For any bearing question, sketch the north lines first and mark the angles. When the answer is a back bearing, check that it differs from the forward bearing by exactly 180°.