Lesson - Circle theorems

Year: Year 11

This lesson is Higher tier and rewards clear reasoning. In the exam you must find the angle and give a reason using the correct theorem name. Draw the circle and write each angle you find on the diagram.

Starter activity

Draw a circle with a compass and mark two points A and B on it. Now mark three different points C, D, E on the major arc and measure angle ACB, ADB and AEB with a protractor. What do you notice? Next, join A and B to the centre and measure angle AOB. How does it compare with your first three angles? Write down your conjecture.

Parts of a circle

Definition: centre, radius, chord, diameter and tangent

The **centre** is the middle point and any line from the centre to the circle is a **radius**. A **chord** joins two points on the circle. A **diameter** is a chord through the centre. A **tangent** touches the circle at one point.

Definition: arc and segment

An **arc** is part of the circumference, and a **segment** is the region between a chord and an arc.

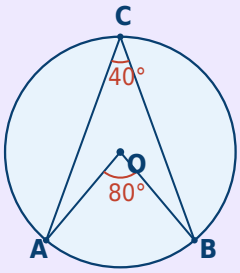
Key fact: isosceles triangles

A triangle formed by two radii and a chord is isosceles, because the two radii are equal, so its base angles are equal.

The theorems

Theorem: angle at the centre

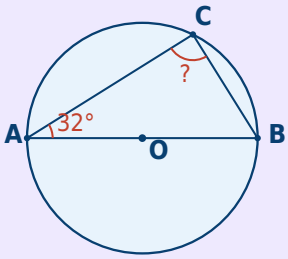
The angle at the centre is twice the angle at the circumference when both stand on the same arc. If the arc subtends 80° at the centre it subtends 40° at the circumference.



Not drawn accurately

Theorem: angle in a semicircle

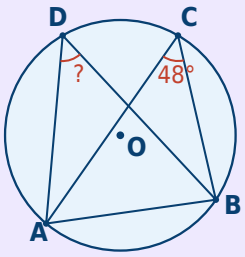
The angle in a semicircle is 90°. If AB is a diameter and C is on the circle, then angle $ACB = 90^\circ$.



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Theorem: angles in the same segment

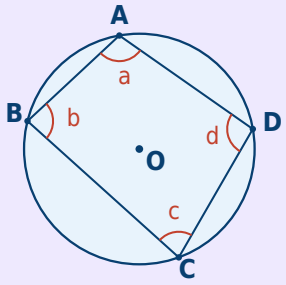
Angles at the circumference standing on the same arc are equal.



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Theorem: cyclic quadrilateral

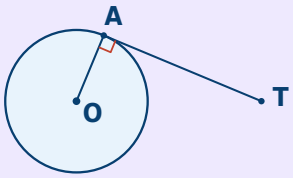
Opposite angles of a cyclic quadrilateral (all four vertices on the circle) add up to 180°. An exterior angle equals the interior opposite angle.



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Theorem: tangent and radius

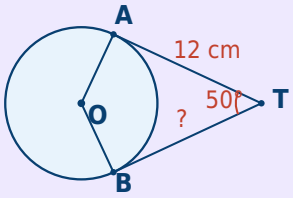
A tangent is perpendicular to the radius at the point of contact.



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Theorem: two tangents

Tangents drawn to a circle from the same point are equal in length, so the triangle formed with the chord is isosceles.



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Theorem: alternate segment

The angle between a tangent and a chord equals the angle in the alternate segment (the segment on the other side of the chord).

Theorem: chord and perpendicular

The perpendicular from the centre to a chord bisects the chord.

Worked examples

Worked example 1: angle at the centre

The angle at the circumference is 64° on arc AB. Find angle AOB and angle OAB.

The angle at the centre is 128°. Triangle OAB is isosceles ($OA = OB$), so $\angle OAB = \dfrac{180 - 128}{2} = 26^\circ$.

Worked example 2: cyclic quadrilateral with algebra

Opposite angles of a cyclic quadrilateral are $(2x + 10)^\circ$ and $(3x - 5)^\circ$. Find x.

$2x + 10 + 3x - 5 = 180 \Rightarrow 5x = 175 \Rightarrow x = 35$. The angles are 80° and 100°.

Worked example 3: tangents

TA and TB are tangents from T and angle $ATB = 50^\circ$. Find $\angle TAB$, $\angle OAB$ and $\angle AOB$.

Since $TA = TB$, the triangle TAB is isosceles and $\angle TAB = \dfrac{180 - 50}{2} = 65^\circ$. Also $\angle OAT = 90^\circ$, so $\angle OAB = 90 - 65 = 25^\circ$. In the kite $OATB$, $\angle AOB = 360 - 90 - 90 - 50 = 130^\circ$.

Worked example 4: chord and radius

A circle has radius 13 cm and a chord of length 24 cm. How far is the chord from the centre?

The perpendicular from the centre bisects the chord, giving a right-angled triangle with hypotenuse 13 and one side 12. The distance is $\sqrt{13^2 - 12^2} = 5$ cm.

Worked example 5: alternate segment

A tangent at A makes angles of 70° with chord AB and 50° with chord AC on the other side. Find the angles of triangle ABC.

By the alternate segment theorem $\angle ACB = 70^\circ$ and $\angle ABC = 50^\circ$. The angles at A on the straight line are $70 + x + 50 = 180$, so $\angle BAC = 60^\circ$, and $70 + 50 + 60 = 180$ checks the triangle.

Proof and method

Proof: angle in a semicircle

Let AB be a diameter with centre O and let C be on the circle. Join OC. Then $OA = OB = OC$ (radii), so triangles OAC and OBC are isosceles. Let $\angle OAC = \angle OCA = x$ and $\angle OBC = \angle OCB = y$. The angles of triangle ABC add up to $180°$: $x + y + (x + y) = 180$, so $x + y = 90$. Therefore angle $ACB = x + y = 90°$.

Method: circle problems

1. Look for radii (isosceles triangles), diameters and tangents first.
2. Mark every angle you can find on the diagram.
3. Write each statement as "angle = value, because ...".
4. Check that angles in a triangle sum to $180°$ and angles on a line sum to $180°$.

Watch out: common mistakes

- Using the centre theorem when the angle is not at the centre.
- Forgetting the reflex angle at the centre: a reflex angle $220°$ gives $110°$ at the circumference on the minor arc.
- Giving a vague reason like "circle rule".
- Assuming the quadrilateral is cyclic when its vertices are not all on the circle.

Worked example 6: Using the tangent-radius angle

TA is a tangent at A to a circle with centre O. Angle $OTA = 24°$. Find angle AOT.

The radius OA meets the tangent at $90°$, so angle $OAT = 90°$. The angles in triangle OAT add to $180°$: $AOT = 180 - 90 - 24 = 66°$. Always write the reason "angle between tangent and radius is $90°$" next to your working.

Worked example 7: Two tangents and a kite

TA and TB are tangents to a circle with centre O. Angle $AOB = 116°$. Find angle ATB.

$OATB$ is a kite with right angles at A and B. The angles of a quadrilateral add to $360°$: $ATB = 360 - 90 - 90 - 116 = 64°$. Alternatively, triangle TAB is isosceles and angle $TAB = \frac{1}{2}(180 - 64) = 58°$, which equals $90° - 32°$, where $32°$ is angle OAB from the isosceles triangle OAB ($180 - 116 = 64$, $64 \div 2 = 32$).

Worked example 8: Angles in the same segment

A, B, C, D lie on a circle. Angle $ADB = 48°$ and angle $DBC = 30°$. Find angle ACB and angle ACD if AC is a diameter.

ACB and ADB are in the same segment on chord AB, so $ACB = 48°$. Since AC is a diameter, angle $ADC = 90°$ and angle $ABC = 90°$. In triangle ABC the third angle is $BAC = 180 - 90 - 48 = 42°$. This is a typical exam chain: one equal-angle step, then

a right angle step.

Worked example 9: A circle theorem with algebra

In a cyclic quadrilateral $ABCD$, angle $A = (3x + 10)^\circ$, angle $B = (2x + 30)^\circ$ and angle $C = (5x - 10)^\circ$. Find x and angle D.

Opposite angles A and C add to 180°: $3x + 10 + 5x - 10 = 180$, so $8x = 180$ and $x = 22.5$. Then $A = 77.5^\circ$, $B = 75^\circ$ and $C = 102.5^\circ$. Angle $D = 180 - B = 105^\circ$. Check: $77.5 + 75 + 102.5 + 105 = 360$.

Worked example 10: Perpendicular from the centre to a chord

A chord AB of length 10 cm is drawn in a circle of radius 13 cm. How far is the chord from the centre?

The perpendicular from the centre bisects the chord, so half the chord is 5 cm. By Pythagoras the distance is $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$ cm. Notice the 5, 12, 13 Pythagorean triple.

Worked example 11: Finding the angle at the centre in a quadrilateral

A, B, C are on a circle, centre O, and angle $ABC = 70^\circ$. Find the reflex angle AOC and angle OAC.

The angle at the centre standing on the same arc as B is $2 \times 70 = 140^\circ$, so the non-reflex angle $AOC = 140^\circ$ and the reflex angle is $360 - 140 = 220^\circ$. Triangle OAC is isosceles with $OA = OC$, so $OAC = (180 - 140) \div 2 = 20^\circ$.

Worked example 12: Alternate segment with a tangent from an external point

TA and TB are tangents to a circle. C is on the major arc AB. Angle $ATB = 48^\circ$. Find angle ACB and give reasons.

Triangle TAB is isosceles because $TA = TB$ (tangents from a point), so angle $TAB = (180 - 48) \div 2 = 66^\circ$. By the alternate segment theorem the angle between the tangent TA and the chord AB equals the angle in the alternate segment, so angle $ACB = 66^\circ$. A second method uses the centre: $AOB = 180 - 48 = 132^\circ$ from the kite, and $ACB = 132 \div 2 = 66^\circ$. Both methods agree, which is a good way to check your answer.

Note: how to write a reason

In circle theorem questions the marks are for the *reason* as much as for the angle. Use short standard phrases: "angle at the centre is twice the angle at the circumference", "angle in a semicircle is 90°", "angles in the same segment are equal", "opposite angles of a cyclic quadrilateral add to 180°", "angle between tangent and radius is 90°

", "tangents from a point are equal in length" and "alternate segment theorem". Write the reason next to the angle, and add an isosceles triangle reason ("radii are equal") whenever you use the centre.

Watch out: spotting the wrong theorem

- The angle at the centre rule only works for angles standing on the **same arc**. Check both angles are made from the same chord.
- A cyclic quadrilateral has all four vertices on the circle; opposite angles add to 180°, not adjacent angles.
- The alternate segment angle is on the **other side** of the chord from the angle with the tangent.
- Do not assume that a line through the centre is a diameter unless both ends touch the circle.

Did you know?

Circle theorems were already known to the ancient Greeks. Thales of Miletus, who lived about 600 BC, is said to have proved that the angle in a semicircle is a right angle, and Euclid collected the other theorems in his book *Elements* around 300 BC. Craftspeople still use the semicircle theorem: a carpenter checks that a corner is square by placing a set square so that its corner touches a circle drawn on its hypotenuse.

Otto's tip

When you see a circle, ask "where is the centre, and which radii can I draw?" Equal radii give isosceles triangles, and these are the key to most circle theorem questions.