Lesson - Units, estimation, rounding and bounds

Year: Year 10

This topic covers measuring in the right units, rounding sensibly, estimating answers to check a calculation, and describing how accurate a rounded number is. Foundation and Higher students both need units, rounding, estimation and error intervals. Upper and lower bounds in calculations are Higher tier only.

Starter activity

Measure the width of your desk with a ruler marked in millimetres and write the measurement. Then measure it again with a ruler marked only in centimetres and write the answer to the nearest centimetre. Are the two numbers the same? What is the largest difference possible between the true width and each of your two answers? Explain why a measurement of “120 cm” could be anywhere between two values, and say what those values are.

Metric units and conversions

Key fact: metric conversions

Quantity	Conversions
Length	1 km = 1000 m; 1 m = 100 cm; 1 cm = 10 mm
Mass	1 tonne = 1000 kg; 1 kg = 1000 g
Capacity	1 litre = 1000 ml; 1 litre = 1000 cm^3
Area	1 m^2 = 10 000 cm^2
Volume	1 m^3 = 1 000 000 cm^3 = 1000 litres

For area and volume the conversion factor is squared or cubed. 2.4 m^2 is $2.4 \times 10\,000 = 24\,000$ cm^2. Speed conversions: 72 km/h $= 72 \times 1000 \div 3600 = 20$ m/s.

Worked example 1

Change 2.4 m^2 into cm^2.

1 m $= 100$ cm, so 1 $\text{m}^2 = 100 \times 100 = 10\,000$ cm^2. $2.4 \times 10\,000 = 24\,000$ cm^2.

Method: converting units

1. Write the conversion factor for the unit (squared for area, cubed for volume).
2. Multiply to go to a smaller unit, divide to go to a bigger unit.

3. Check that the size of the answer makes sense.

Rounding

- **Decimal places (d.p.):** count digits after the decimal point. 4.7863 to 2 d.p. is 4.79, because the next digit (6) is 5 or more.
- **Significant figures (s.f.):** the first significant figure is the first non-zero digit. 0.004582 to 2 s.f. is 0.0046. $58\,730$ to 3 s.f. is $58\,700$: use zeros as place holders.
- Look only at the digit after the one you are rounding to. Do not round in stages.

Worked example 2

Round 0.004582 to 2 significant figures.

The first significant figure is the 4. The second is the 5. The next digit is 8, so round up: 0.0046.

Estimating

Method: estimating

To estimate, round each number to 1 significant figure, then calculate with the easy numbers. $\dfrac{48.2 \times 5.1}{0.52} \approx \dfrac{50 \times 5}{0.5} = 500$. Always show the rounded values: the method mark is for them. If every number in a product is rounded up, the estimate is an *overestimate*; if the denominator is rounded up the estimate is an underestimate.

Worked example 3

Estimate $\dfrac{48.2 \times 5.1}{0.52}$.

Round each number to 1 s.f.: $\dfrac{50 \times 5}{0.5} = \dfrac{250}{0.5} = 500$.

Error intervals

Definition: error interval

An **error interval** shows the range of true values that round to a given number. It is written $a \leq x < b$, where a is the lower bound and b is the upper bound.

Rule: error intervals

A rounded number stands for a range of true values. The **error interval** is written $a \leq x$, where the lower bound is included and the upper bound is not.

- $x = 3.4$ to 1 d.p.: $3.35 \leq x$.
- $n = 60$ to the nearest 10: $55 \leq n$.
- $m = 8$ kg to the nearest kg: $7.5 \leq m$.
- $x = 0.35$ to 2 s.f.: $0.345 \leq x$.

Method: halve the rounding unit (the degree of accuracy), subtract it for the lower bound and add it for the upper bound. For 1 d.p. the unit is 0.1, so half is 0.05. The upper bound is not reached: use "<", never "≤". If a number is truncated instead of rounded, the interval is the whole unit: 8.2 truncated to 1 d.p. gives $8.2 \leq x$.

Worked example 4

$x = 3.4$ to 1 decimal place. Write down the error interval for x.

The rounding unit is 0.1, so half of it is 0.05. Lower bound $3.4 - 0.05 = 3.35$. Upper bound $3.4 + 0.05 = 3.45$. $3.35 \leq x < 3.45$.

Upper and lower bounds in calculations (Higher tier)

Definition: upper and lower bounds

The **lower bound** is the smallest value that rounds to the number and the **upper bound** is the value it just stops being. For 8 kg to the nearest kg they are 7.5 and 8.5.

Rule: bounds in calculations (Higher tier)

Write the bounds of every measurement, then choose them to make the answer as large or as small as possible.

- **Add:** upper + upper for the maximum; lower + lower for the minimum.
- **Subtract:** $a - b$ is largest with the upper bound of a and the lower bound of b; smallest with the lower bound of a and the upper bound of b.
- **Multiply:** upper × upper for the maximum; lower × lower for the minimum.
- **Divide:** $a \div b$ is largest with upper a and lower b; smallest with lower a and upper b.

Worked example 5

A rectangle is 12 cm by 8 cm, each to the nearest cm.

Upper bound of area $= 12.5 \times 8.5 = 106.25$ cm^2. Lower bound of perimeter $= 2(11.5 + 7.5) = 38$ cm.

Worked example 6

Speed = distance ÷ time with distance 100 m (nearest m) and time 12.5 s (nearest 0.1 s).

Upper bound of speed $= \dfrac{100.5}{12.45} = 8.07$ m/s to 3 s.f.

Common mistakes

Watch out

- Forgetting that the area conversion is $10\,000$, not 100.
- Using the wrong inequality symbols in an error interval. The upper bound has "<".
- Rounding to 1 s.f. one number but not another in an estimate, or using 3.14 for π when an estimate needs $\pi \approx 3$.
- Using the upper bound of the denominator to find the upper bound of a quotient.

Measuring and approximating

Almost every number in real life is an approximation. A ruler can only be read to the nearest millimetre, a clock to the nearest second, and a population count is out of date as soon as it is finished. Rounding makes the number simpler and tells the reader how much accuracy to trust. There are two ways to describe accuracy: decimal places count digits after the decimal point, and significant figures count digits starting from the first non-zero digit. When a value has been rounded, the original number lies in an error interval, whose ends are the lower bound and the upper bound. The lower bound is always included, and the upper bound is not included in the inequality, written $a \le x < b$. To find the bounds, half the unit of accuracy is added and subtracted: 5.6 to 1 decimal place lies between 5.55 and 5.65.

Worked example 7: bounds of a sum and a product

$p = 4.2$ and $q = 3.1$, both to 1 d.p. Find the greatest possible value of $p + q$ and of pq [3 marks].

The upper bounds are 4.25 and 3.15. The greatest sum is $4.25 + 3.15 = 7.40$. The greatest product is $4.25 \times 3.15 = 13.3875$. Notice that you use upper bounds for both numbers because the product is increasing in each. If one number is subtracted or divided by, the other bound must be used.

Worked example 8: bounds of a quotient

A car travels 150 m (to the nearest 10 m) in 8.0 s (to 1 d.p.). Find the bounds of the speed [4 marks].

Distance: $145 \le d < 155$. Time: $7.95 \le t < 8.05$. For the greatest speed use the greatest distance and the least time: $\dfrac{155}{7.95} = 19.50$ m/s. For the least speed use the least distance and the greatest time: $\dfrac{145}{8.05} = 18.01$ m/s. The speed is between 18.0

and 19.5 m/s.

Worked example 9: choosing the accuracy of an answer

Both bounds of a quantity are 4.836 and 4.842. How should the answer be quoted? [2 marks]

To 3 significant figures the bounds are 4.84 and 4.84, which agree. To 4 significant figures they are 4.836 and 4.842, which differ. So the answer can be quoted as 4.84 to 3 significant figures, but not to 4. This is the rule: quote the answer to the greatest accuracy at which both bounds round to the same value.

Worked example 10: a conversion with two steps

A swimming pool holds 90 m³ of water. How many litres is this, and how long does it take to fill at 300 litres per minute? [3 marks]

$1\ \text{m}^3 = 1000$ litres, so $90\ \text{m}^3 = 90\,000$ litres. Time $= \dfrac{90\,000}{300} = 300$ minutes, which is 5 hours. Estimating gives a quick check: $100\,000 \div 300 \approx 333$ minutes, close to 300.

Note: how to estimate

To estimate, round each number to 1 significant figure, then do the calculation using easy numbers. Write the rounded values on a line of working, because marks are given for them. Division by a number between 0 and 1 makes the answer bigger, which is a common source of mistakes when you check an estimate.

Watch out: upper bounds in subtraction and division

To find the greatest value of $a - b$ use the upper bound of a and the *lower* bound of b. To find the greatest value of $\dfrac{a}{b}$ use the upper bound of a and the *lower* bound of b. Mixing them up is the most common error in a bounds question.

Did you know?

The metric system was introduced in France after the Revolution in 1795 and was designed to be based on nature: the metre was defined as one ten-millionth of the distance from the North Pole to the equator. The UK still uses a mixture of units, with miles on the roads, pints in the pub and kilograms in the supermarket. In 1999 a spacecraft called the Mars Climate Orbiter, which cost 125 million dollars, was lost because one team used pounds-force and the other used newtons. A simple unit error caused it to enter Mars' atmosphere too low and burn up. This is why exam questions are so careful about units, and why professional engineers always write the unit after every number.

Otto's tip

For error intervals, draw a little number line. Mark the rounded value in the middle and half a unit either side. For bounds in a calculation, write 'max' and 'min' above the numbers you use so you pick the right ends.

Always give a final answer to a sensible degree of accuracy and include the units. Do not round in the middle of a multi-step calculation: keep full calculator values and round at the end.