Lesson - Surds and exact values

Year: Year 10

An exact value is one that has not been rounded. At GCSE you give exact answers in terms of π, as fractions, and (Higher tier only) in surd form. Exact answers are asked for on non-calculator papers and whenever a question says "give your answer in terms of π" or "leave your answer in surd form".

Starter activity

Use a calculator to find $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{16}$. Which of these displays seems to stop? Write down the first ten digits of $\sqrt{2}$. Do they repeat in a pattern? Now draw a square of side 1 cm and measure its diagonal as accurately as you can. Why is it impossible to write the exact length as a fraction or terminating decimal? What could you write instead so that the answer is exact?

Answers in terms of π (Foundation and Higher)

Definition: exact value

An **exact value** has not been rounded. It may be a fraction, a multiple of π or (Higher tier) a surd such as $3\sqrt{5}$.

Key fact: answers in terms of pi

Do not multiply by 3.14 or press the π key. Treat π like a letter in algebra and leave it in the answer.

- Circumference $C = \pi d = 2\pi r$. A circle with diameter 10 cm has $C = 10\pi$ cm.
- Area $A = \pi r^2$. A circle with radius 6 cm has $A = 36\pi$ cm^2.
- Volume of a cylinder $V = \pi r^2 h$. With $r = 3$ and $h = 5$: $V = 45\pi$ cm^3.

Always square the radius before multiplying: $(6)^2 = 36$, so 36π, not 6π or 12π.

What is a surd? (Higher tier)

Definition: surd (Higher tier)

A surd is a root that cannot be written as a fraction because it is irrational, such as $\sqrt{2}$, $\sqrt{3}$ or $\sqrt{10}$. Roots of perfect squares, such as $\sqrt{16} = 4$, are not surds. A surd has an infinite, non-repeating decimal, so keeping it in surd form is the only way to be exact.

Rules for surds (Higher tier)

Rule: the rules for surds (Higher tier)

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$
- $\sqrt{a} \times \sqrt{a} = a$
- $\sqrt{a} + \sqrt{b}$ is not $\sqrt{a+b}$: you cannot add under one root.

Simplifying surds (Higher tier)

Method: simplifying surds (Higher tier)

Find the largest square number that is a factor, split it off, then take its root. $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$. Learn the square numbers 4, 9, 16, 25, 36, 49 and 100 so you spot them quickly. Another example: $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$.

Method: steps for simplifying a surd (Higher tier)

1. List the square numbers 4, 9, 16, 25, 36, 49, 100.
2. Find the largest one that divides into the number under the root.
3. Split the root: $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.
4. Take the root of the square number.

Adding and subtracting surds (Higher tier)

Rule: adding and subtracting surds (Higher tier)

You can only combine "like" surds, which have the same number under the root, just as you combine like terms: $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$. Simplify each surd first. $\sqrt{18} + \sqrt{8} = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$, and $\sqrt{75} - \sqrt{27} = 5\sqrt{3} - 3\sqrt{3} = 2\sqrt{3}$.

Multiplying and expanding (Higher tier)

Rule: multiplying and expanding surds (Higher tier)

Multiply the whole numbers together and the surds together: $2\sqrt{5} \times 3\sqrt{5} = 6 \times 5 = 30$. To expand brackets use FOIL (or a grid) and then simplify. $(2+\sqrt{3})^2 = 4 + 2\sqrt{3} + 2\sqrt{3} + 3 = 7 + 4\sqrt{3}$. Brackets of the form $(a+\sqrt{b})(a-\sqrt{b})$

give the integer $a^2 - b$, because the surd terms cancel: $(3+\sqrt{2})(3-\sqrt{2}) = 9 - 2 = 7$.

Rationalising the denominator (Higher tier)

Definition: rationalising and conjugate (Higher tier)

Rationalising the denominator means removing the surd from the bottom of a fraction. The **conjugate** of $a + \sqrt{b}$ is $a - \sqrt{b}$.

Method: rationalising the denominator (Higher tier)

A fraction is not in its simplest surd form if there is a surd in the denominator. Multiply the top and bottom by the surd to remove it. $\dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$.

For a denominator such as $3 + \sqrt{5}$, multiply top and bottom by the **conjugate** $3 - \sqrt{5}$, using $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$. $\dfrac{4}{3+\sqrt{5}} = \dfrac{4(3-\sqrt{5})}{9-5} = \dfrac{4(3-\sqrt{5})}{4} = 3 - \sqrt{5}$.

Surds in geometry

Key fact: exact lengths from Pythagoras

Exact lengths often come from Pythagoras' theorem. In a right-angled triangle with shorter sides 6 and 3, the hypotenuse is $\sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$. The side of a square of area 48 is $\sqrt{48} = 4\sqrt{3}$, so its perimeter is $16\sqrt{3}$.

More worked examples in exam style

Worked example 1 (exact circle problem)

A circle has circumference 20π cm. Find its area in terms of π [3 marks].

$2\pi r = 20\pi$, so $r = 10$. $A = \pi \times 10^2 = 100\pi$ cm^2.

Worked example 2 (Higher tier)

Show that $\dfrac{\sqrt{12}+\sqrt{27}}{\sqrt{3}} = 5$ [3 marks].

$\sqrt{12} = 2\sqrt{3}$ and $\sqrt{27} = 3\sqrt{3}$. The numerator is $5\sqrt{3}$. Divide by $\sqrt{3}$: 5.

Worked example 3 (Higher tier)

Solve $x\sqrt{2} = 10$, giving x in the form $a\sqrt{b}$ [2 marks].

$x = \frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$.

Worked example 4 (Higher tier)

Triangle ABC has a right angle at B, $AB = \sqrt{7}$ cm and $BC = 3$ cm. Find AC [2 marks].

$AC^2 = 7 + 9 = 16$, so $AC = 4$ cm. Notice the answer is an integer, so it is not a surd.

Common mistakes

Watch out

- Writing $\sqrt{9 + 16} = \sqrt{9} + \sqrt{16}$. It is $\sqrt{25} = 5$, not 7.
- Stopping at $\sqrt{50} = \sqrt{10} \times \sqrt{5}$, which does not simplify. Use the largest square factor.
- Forgetting the middle terms when squaring a bracket: $(2 + \sqrt{3})^2$ is not $4 + 3$.
- Rationalising by multiplying only the top or only the bottom.
- Rounding too early in a question that asks for an exact answer.

Exam technique

Exam technique

- "Exact" means no decimals. Check that your final answer has no rounded numbers.
- Show every line when rationalising, including the multiplication of the numerator and of the denominator.
- Use your calculator to check: $\sqrt{50} = 7.0710\ldots$ and $5\sqrt{2} = 7.0710\ldots$.
- If a question says "in the form $a\sqrt{b}$", make sure b has no square factor left.

Why exact values matter

Calculators give decimal answers, but many numbers in geometry cannot be written exactly as a decimal. The diagonal of a unit square is $\sqrt{2}$, the circumference of a circle is a multiple of π, and the height of an equilateral triangle involves $\sqrt{3}$. If you round at the beginning of a calculation, small errors build up, so it is better to keep numbers in exact form such as $5\sqrt{2}$ or 36π until the very last line. Exam questions that say “give your

answer in terms of π" or "in surd form" expect an exact answer and give no marks for a rounded decimal. At Foundation tier the most important exact form is the multiple of π, and at Higher tier you must also be able to simplify, add, multiply and rationalise surds. These skills connect directly to Pythagoras' theorem, trigonometry and solving quadratic equations with the formula.

Worked example 5: simplifying a more complicated surd

Write $\sqrt{450}$ in the form $a\sqrt{b}$ where b is as small as possible [2 marks].

Find the largest square factor of 450. $450 = 225 \times 2$ and $225 = 15^2$, so $\sqrt{450} = \sqrt{225} \times \sqrt{2} = 15\sqrt{2}$. If you do not spot 225 immediately, go in stages: $450 = 9 \times 50$, so $\sqrt{450} = 3\sqrt{50} = 3 \times 5\sqrt{2} = 15\sqrt{2}$. Both routes give the same answer.

Worked example 6: the exact area of a sector

A sector has radius 9 cm and angle 80°. Find the area in terms of π [2 marks].

Area $= \dfrac{80}{360} \times \pi \times 81 = \dfrac{2}{9} \times 81\pi = 18\pi$ cm². Simplify the fraction first: $\dfrac{80}{360} = \dfrac{2}{9}$. Then $\dfrac{2}{9} \times 81 = 18$. The exact area is 18π cm², which is 56.5 cm² to 1 decimal place.

Worked example 7: the conjugate method

Express $\dfrac{5}{\sqrt{7} - \sqrt{2}}$ in the form $a(\sqrt{7} + \sqrt{2})$ [3 marks].

Multiply top and bottom by the conjugate $\sqrt{7} + \sqrt{2}$. The denominator becomes $(\sqrt{7} - \sqrt{2})(\sqrt{7} + \sqrt{2}) = 7 - 2 = 5$. The numerator is $5(\sqrt{7} + \sqrt{2})$. So the fraction is $\dfrac{5(\sqrt{7} + \sqrt{2})}{5} = \sqrt{7} + \sqrt{2}$ and $a = 1$. Check numerically: $\dfrac{5}{2.6458 - 1.4142} = 4.06$ and $\sqrt{7} + \sqrt{2} = 4.06$ ✓.

Worked example 8: surd in a context

A square has area 50 cm². Find its side and perimeter in exact form, then compare with a circle of the same area [4 marks].

Side $= \sqrt{50} = 5\sqrt{2}$ cm and perimeter $= 4 \times 5\sqrt{2} = 20\sqrt{2} = 28.3$ cm. A circle of area 50 has $\pi r^2 = 50$, so $r = \sqrt{50/\pi} = 3.99$ cm and circumference $2\pi r = 25.1$ cm. The circle has a shorter perimeter, which is why circles enclose the largest area for a given perimeter.

Note: rules for surds

The rules are $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ and $(\sqrt{a})^2 = a$. Note that $\sqrt{a+b}$ is *not* $\sqrt{a} + \sqrt{b}$: $\sqrt{9+16} = 5$ but $\sqrt{9} + \sqrt{16} = 7$. Surds can be added only when they have the same surd part, so $2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$ but $\sqrt{2} + \sqrt{3}$ cannot be simplified.

Watch out: rounding too early

If you round $\sqrt{2}$ to 1.41 at the start of a calculation, your final answer can be wrong in the second decimal place. Keep the exact form or use your calculator's memory.

Did you know?

The discovery that some numbers cannot be written as fractions is traced to the ancient Greek school of Pythagoras, around 500 BC. The Pythagoreans believed that everything in the universe could be described by whole numbers and their ratios. According to legend, a member of the group named Hippasus showed that the diagonal of a unit square, $\sqrt{2}$, cannot be a ratio of whole numbers, and the shocked brotherhood was so upset that Hippasus was thrown off a ship. Whether or not the story is true, irrational numbers shaped mathematics: $\sqrt{2}$, π and e all have infinite non-repeating decimals. In 2019 computers calculated π to over 31 trillion digits, but every circle on Earth can be measured to atomic accuracy with fewer than 40.

Otto's tip

When you see an exact-value question, put the calculator away. Simplify each surd on its own first, then combine. Check your work with a calculator afterwards: both forms should give the same decimal.