Lesson - Simultaneous equations

Year: Year 10

Simultaneous equations are two (or more) equations that are true at the same time. Solving them means finding the values of the unknowns that satisfy every equation. At GCSE you solve pairs of linear equations by elimination, by substitution and graphically, and form equations from problems. Solving a linear equation together with a quadratic equation is Higher tier only.

Starter activity

Maya buys 2 pens and 1 pencil for 70p. Tom buys 1 pen and 1 pencil for 45p. Without writing any algebra, work out the difference between Maya's and Tom's purchase. What does that difference tell you about the price of one pen? Then find the price of a pencil. Now write the same problem with p for a pen and q for a pencil and see which steps of your reasoning correspond to elimination.

Elimination

Definition: simultaneous equations

Simultaneous equations are two or more equations that are true at the same time. A solution gives a value for each unknown that works in every equation.

Definition: elimination

Elimination means making the coefficients of one letter match, then adding or subtracting the equations so that the letter disappears.

Method: elimination

1. Label the equations (1) and (2).
2. Multiply one or both so that one letter has matching coefficients.
3. Same signs: subtract. Different signs: add.
4. Solve for the remaining letter, substitute back, and check in the other equation.

Rule: elimination

Make the coefficients of one letter match, then add or subtract the equations to eliminate it.

- **Same signs:** subtract. **Different signs:** add. (Think "SSS": same signs subtract.)

Worked example 1

$x + y = 10$ **and** $x - y = 4$**.**

Add: $2x = 14$, so $x = 7$. Then $7 + y = 10$, $y = 3$.

Worked example 2

$3x + 2y = 12$ **and** $5x - 3y = 1$**.**

The coefficients of y are 2 and -3, so multiply the first by 3 and the second by 2 to get $9x + 6y = 36$ and $10x - 6y = 2$. Add: $19x = 38$, so $x = 2$. Substitute: $6 + 2y = 12$, $y = 3$. **Check** in the other equation: $10 - 9 = 1$. Correct.

Worked example 3

$4x + 3y = 18$ **and** $2x + 5y = 16$**.**

Multiply the second by 2: $4x + 10y = 32$. Subtract: $7y = 14$, $y = 2$. Then $4x + 6 = 18$, $x = 3$.

Substitution

Method: substitution

1. Make one letter the subject of one equation (or use an equation that already has it).
2. Replace that letter in the other equation, using brackets.
3. Solve, then substitute back to find the second letter.

Rule: substitution

Use this when one equation already has a letter as the subject. $2x + 3y = 13$ and $y = 4x - 5$. Replace y: $2x + 3(4x - 5) = 13$, so $14x - 15 = 13$, $x = 2$, and $y = 4 \times 2 - 5 = 3$. Substitution is also the method used for linear/quadratic pairs.

Solving graphically

Rule: solving graphically

Draw both lines. The solution is the coordinates of the point where the lines cross. For $y = x + 1$ and $x + y = 7$ the lines meet at $(3, 4)$, so $x = 3$, $y = 4$. If the lines are parallel (same gradient, different intercepts) there is no solution. The equations $x + y = 5$ and $2x + 2y = 7$ have no solution: the second is $x + y = 3.5$, parallel to the first.

Forming simultaneous equations from problems

Method: forming simultaneous equations from problems

1. Define two letters.
2. Write two equations from the information.
3. Solve them, then answer in context with units.

Three pens and two notebooks cost £5.50. Two pens and five notebooks cost £11. Let p be a pen and n a notebook in pounds: $3p + 2n = 5.5$, $2p + 5n = 11$. Multiply: $15p + 10n = 27.5$ and $4p + 10n = 22$. Subtract: $11p = 5.5$, $p = 0.5$, then $n = 2$. A pen costs 50p and a notebook £2.

Two adult tickets and three child tickets cost £28; one adult and four children cost £24. $2a + 3c = 28$ and $a + 4c = 24$. Double the second: $2a + 8c = 48$. Subtract: $5c = 20$, $c = 4$, $a = 8$.

Linear and quadratic pairs (Higher tier)

Method: linear and quadratic pairs (Higher tier)

Substitute the linear equation into the quadratic and solve the resulting quadratic. This usually gives two pairs of solutions.

Worked example 4

$x^2 + y^2 = 25$ **and** $y = x + 1$**.**

$x^2 + (x + 1)^2 = 25$ gives $2x^2 + 2x - 24 = 0$, so $x^2 + x - 12 = 0$, $(x + 4)(x - 3) = 0$. $x = 3$ gives $y = 4$; $x = -4$ gives $y = -3$. The solutions are $(3, 4)$ and $(-4, -3)$. Geometrically, the line cuts the circle at two points.

Worked example 5

$y = x^2 - 5$ and $y = x + 1$: $x^2 - x - 6 = 0$, $x = 3$ or -2, giving $(3, 4)$ and $(-2, -1)$.

More worked examples in exam style

Worked example 6

A café sells 2 teas and 3 coffees for £9.00 and 3 teas and 1 coffee for £6.50. Find the price of a tea and of a coffee [4 marks].

$2t + 3c = 9$, $3t + c = 6.5$. Multiply the second by 3: $9t + 3c = 19.5$. Subtract the first: $7t = 10.5$, so $t = 1.5$. Then $c = 6.5 - 4.5 = 2$. A tea costs £1.50 and a coffee £2.00. Check in the first equation: $3 + 6 = 9$.

Worked example 7

Solve $2x + 5y = 1$ and $3x - 2y = 11$ [4 marks].

Multiply the first by 2 and the second by 5: $4x + 10y = 2$ and $15x - 10y = 55$. Add: $19x = 57$, $x = 3$. Then $6 + 5y = 1$, $y = -1$.

Worked example 8 (Higher tier)

Solve $y = 2x + 1$ and $y = x^2 - 2$ [4 marks].

$x^2 - 2 = 2x + 1$, so $x^2 - 2x - 3 = 0$, $(x - 3)(x + 1) = 0$. $x = 3$, $y = 7$ or $x = -1$, $y = -1$.

Common mistakes

Watch out

- Adding when you should subtract (watch the signs of the terms you are eliminating).
- Multiplying only one term of an equation, not every term.
- Finding x and forgetting to find y.
- Pairing the values wrongly in a linear/quadratic pair: always substitute back into the *linear* equation.
- Not checking the answer in the other equation.

Exam technique

Exam technique

- Number your equations and show the multiplier used.
- Check the solution in *both* original equations.
- In problem questions, finish with a sentence giving the answer with units.

Two unknowns, two equations

One equation with two unknowns, such as $x + y = 10$, has infinitely many solutions: $(1, 9)$, $(2, 8)$, $(3, 7)$ and so on, all on a straight line. A second equation gives a second line, and the solution is the single point where the two lines cross. Algebra finds that point exactly. In elimination you add or subtract the equations to remove one letter, after multiplying if necessary so that a pair of coefficients matches. In substitution you rewrite one equation to make a letter the subject, and replace that letter in the other equation. Either method works for any pair, so choose the one that needs fewer steps: use substitution if one equation already has the form $y = \ldots$ or $x = \ldots$, and elimination if both are in the form $ax + by = c$. Always check your solution in the equation you did not use.

Worked example 9: the sign rule for elimination

Solve $5x + 3y = 19$ and $2x + 3y = 10$ [3 marks].

The y coefficients are the same, with the same sign, so *subtract* the second equation from the first: $3x = 9$, so $x = 3$. Substitute into the second: $6 + 3y = 10$, $3y = 4$, $y = \frac{4}{3}$. Check in the first: $15 + 4 = 19$ ✓. If the signs had been different, such as $+3y$ and $-3y$, you would have added.

Worked example 10: a worded problem with fractions

Three pens and two rulers cost £4.40. Two pens and five rulers cost £5.50. Find the cost of each [4 marks].

Equations: $3p + 2r = 4.40$ and $2p + 5r = 5.50$. Multiply the first by 2 and the second by 3: $6p + 4r = 8.80$ and $6p + 15r = 16.50$. Subtract: $11r = 7.70$, so $r = 0.70$. Then $3p + 1.40 = 4.40$, so $p = 1.00$. A pen costs £1.00 and a ruler 70p. Check: $2 + 3.50 = 5.50$ ✓.

Worked example 11: graphical solution and meaning

Draw $y = x + 2$ and $y = 8 - 2x$ and find the solution [3 marks].

Table for $y = x + 2$: $(0, 2)$, $(2, 4)$, $(4, 6)$. Table for $y = 8 - 2x$: $(0, 8)$, $(2, 4)$, $(4, 0)$. Both lines pass through $(2, 4)$, so the solution is $x = 2$, $y = 4$. Check algebraically: $x + 2 = 8 - 2x$ gives $3x = 6$, $x = 2$. This is a good method of checking graph answers.

Worked example 12: a break-even problem

A baker's costs are $C = 150 + 0.8n$ and her income is $I = 2.5n$ for n loaves. How many loaves must she sell to break even? [3 marks]

Break-even is when $C = I$: $150 + 0.8n = 2.5n$, so $150 = 1.7n$ and $n = 88.2$. She must sell 89 loaves to make a profit, because she cannot sell 0.2 of a loaf. Check: 89 loaves give income 222.50 and costs 221.20, a profit of £1.30.

Note: special cases

If the two lines have the same gradient and different intercepts they are parallel and there is no solution; if they are the same line there are infinitely many solutions. In elimination this shows up as the letters cancelling and leaving a false statement such as $0 = 5$ (no solution) or a true one such as $0 = 0$ (infinitely many).

Watch out: signs when subtracting

When you subtract one equation from another, subtract *every* term, including the right-hand side. For example $(5x - 2y = 11) - (3x - 2y = 5)$ gives $2x = 6$ and $(5x + 2y = 11) - (3x - 2y = 5)$ cannot eliminate y by subtracting but adds to $4y$. Write

down the operation you use at each step.

Did you know?

Simultaneous equations were solved by Chinese mathematicians more than 2000 years ago. The book *Nine Chapters on the Mathematical Art* from about 200 BC describes a method of arranging the coefficients in a table and subtracting rows, which is very close to the Gaussian elimination used by computers today. In the West, the method is named after Carl Friedrich Gauss, who used it in 1809 to find the orbit of the asteroid Pallas. Today, simultaneous equations help engineers design bridges, economists find the point where supply meets demand and computer graphics designers work out where a ray of light hits an object. Solving systems with thousands of unknowns is one of the main jobs of modern computers.

Otto's tip

Always label your equations (1) and (2), and say what you do to them, for example (1) $\times 3$. Finish by substituting into the equation you did not use for the substitution step. If it works, your answer is right.