# Lesson - Sequences

Year: Year 10

A sequence is a list of numbers that follow a rule. At GCSE you continue sequences, use term-to-term and position-to-term rules, find the $n$th term of linear sequences, decide whether a number is in a sequence, and recognise special sequences. Quadratic $n$th terms and geometric sequences with a surd ratio are Higher tier only.

**Starter activity**

Look at the sequence 2, 5, 8, 11, ... and predict the 20th term without writing out all twenty. What do you add each time? How many times would you have added by the 20th term? Write down a rule that works for any position. Now compare with 2, 4, 8, 16, ... Can the same rule work? What changes? Find the 10th term of each by whatever method you like.

## Vocabulary

**Definition: term**

A **term** is one number in the sequence. The first term is $T(1)$ or $u_1$, the $n$th term is the term in position $n$.

**Definition: term-to-term rule**

The **term-to-term rule** tells you how to get from one term to the next, for example "add 4" or "multiply by 2 then add 1".

**Definition: position-to-term rule**

The **position-to-term rule** (the $n$th term) gives a term directly from its position, for example $4n - 1$.

## Arithmetic (linear) sequences

**Definition: arithmetic sequence**

An **arithmetic sequence** goes up or down by the same amount each time. That amount is the **common difference** $d$.

**Rule: the nth term of an arithmetic sequence**

An **arithmetic sequence** goes up or down by the same amount each time, the **common difference** $d$. The $n$th term is $dn + (\text{first term} - d)$. For 5, 8, 11, 14: $d = 3$,

and $5 - 3 = 2$, so the $n$th term is $3n + 2$. For 20, 17, 14, 11: $d = -3$, and $20 - (-3) = 23$, so the $n$th term is $-3n + 23$ (or $23 - 3n$). Check by substituting $n = 1$ and $n = 2$.

If you only know two terms, find $d$ from the difference in terms divided by the difference in positions. If the 2nd term is 9 and the 5th is 21: $d = \frac{21-9}{5-2} = 4$; first term $9 - 4 = 5$; $n$th term $4n + 1$.

**Worked example 1**

**Find the $n$th term of $7,\ 11,\ 15,\ 19,\ \ldots$**

The common difference is 4, so the sequence starts with $4n$. $4 \times 1 = 4$ and the first term is 7, so add 3. The $n$th term is $4n + 3$. Check: $n = 3$ gives $15$.

## Using the $n$th term

**Method: using the nth term**

- The 10th term of $4n - 1$ is $4 \times 10 - 1 = 39$.
- **Is a number in the sequence?** Set the $n$th term equal to it and solve. Is 100 in $3n + 1$? $3n = 99$, $n = 33$, a whole number, so yes. Is 50 in $4n - 3$? $4n = 53$, $n = 13.25$, not whole, so no.
- First term over 100 of $7n - 4$: $7n - 4 > 100$, $n > 14.86$, so $n = 15$, term 101.

**Worked example 2**

**The $n$th term of a sequence is $5n + 2$. Is 62 a term? Is 100 a term?**

For 62: $5n + 2 = 62$, $5n = 60$, $n = 12$, a whole number, so yes. For 100: $5n = 98$, $n = 19.6$, not a whole number, so no.

## Sequences from patterns

**Key fact: sequences from patterns**

Pattern $n$ uses $5n + 1$ matchsticks, so pattern 12 uses 61 and 101 matches are used by pattern $n = 20$. The coefficient of $n$ is the number added each time; the constant is what is left over.

## Other important sequences

**Key fact: special sequences**

- Square numbers: $1, 4, 9, 16, 25$: $n^2$.

- Cube numbers: $1, 8, 27, 64$: $n^3$.
- Triangular numbers: $1, 3, 6, 10, 15, 21$: add 2, then 3, then 4, ...
- Fibonacci-type: each term is the sum of the two before: $1, 1, 2, 3, 5, 8, 13$.
- Powers of 2: $2, 4, 8, 16$. Powers of 10: $10, 100, 1000$.

## Geometric sequences

**Definition: geometric sequence**

A **geometric sequence** is multiplied by the same number each time. That number is the **common ratio** $r$.

**Rule: geometric sequences**

A geometric sequence multiplies by a constant **common ratio** $r$ each time. For 3, 6, 12, 24: $r = 2$, and the $n$th term is $3 \times 2^{n-1}$. The 8th term is $3 \times 2^7 = 384$. Sequences such as 2, 6, 18 have $n$th term $2 \times 3^{n-1}$; the 6th term is $2 \times 243 = 486$.

(Higher) The ratio can be a surd. For $3,\ 3\sqrt{2},\ 6$: $r = \sqrt{2}$. The next terms are $6\sqrt{2}$ and $12$; the 8th term is $3 \times (\sqrt{2})^7 = 3 \times 8\sqrt{2} = 24\sqrt{2}$.

**Worked example 3**

**Find the 7th term of the geometric sequence** $5,\ 10,\ 20,\ \ldots$

The common ratio is 2, so the $n$th term is $5 \times 2^{n-1}$. The 7th term is $5 \times 2^6 = 5 \times 64 = 320$.

## Quadratic sequences (Higher tier)

**Method: quadratic nth terms (Higher tier)**

If the first differences are not constant but the second differences are, the $n$th term is quadratic: $an^2 + bn + c$, where $a$ is half the second difference.

For 3, 8, 15, 24: first differences 5, 7, 9; second differences 2, so $a = 1$. Subtract $n^2$ (1, 4, 9, 16) from the terms: $2, 4, 6, 8$, which is $2n$. So the $n$th term is $n^2 + 2n$.

For 5, 12, 23, 38: differences 7, 11, 15; second differences 4, so $a = 2$. Subtract $2n^2$ (2, 8, 18, 32): $3, 4, 5, 6$, which is $n + 2$. The $n$th term is $2n^2 + n + 2$.

**Worked example 4**

**Find the $n$th term of** $2,\ 7,\ 14,\ 23,\ \ldots$ **(Higher tier)**

First differences: 5, 7, 9. Second differences: 2, 2, so $a = 1$. Subtract $n^2$ (1, 4, 9, 16):

left with $1, 3, 5, 7$, which is $2n - 1$. The $n$th term is $n^2 + 2n - 1$. Check: $n = 4$ gives $16 + 8 - 1 = 23$.

## Common mistakes

**Watch out**

- Giving the term-to-term rule ("add 3") when the $n$th term is requested.
- Writing the $n$th term as $n + 3$ instead of $3n + 2$: check against the first term.
- Saying a number is in the sequence without checking that $n$ is a whole number.
- Squaring the common difference: the difference is multiplied by $n$, not by $n^2$.

## Seeing the pattern

A sequence is an ordered list of numbers that follows a rule. There are two ways to describe the rule. A term-to-term rule tells you how to get from one term to the next, such as “add 4”, and a position-to-term rule, the $n$th term, lets you jump straight to any term, such as $4n + 3$. At GCSE there are three main families. Arithmetic sequences have a constant difference and an $n$th term of the form $dn + c$. Geometric sequences have a constant ratio and terms of the form $ar^{n-1}$. Quadratic sequences have a constant second difference and an $n$th term of the form $an^2 + bn + c$. To identify the family, write down the differences between terms: if they are equal the sequence is arithmetic, if the second differences are equal it is quadratic, and if the terms are multiplied by a constant ratio it is geometric.

**Worked example 5: from a pattern to a formula**

**Pattern 1 has 4 squares in a plus shape, pattern 2 has 7, pattern 3 has 10. Find the number of squares in pattern 20 [3 marks].**

The difference is 3 each time, so the $n$th term is $3n + c$. When $n = 1$, $3 + c = 4$, so $c = 1$. The formula is $3n + 1$. Then the 20th pattern has $3 \times 20 + 1 = 61$ squares. Check the formula with pattern 3: $3 \times 3 + 1 = 10$ ✓.

**Worked example 6: working backwards**

**The 5th term of an arithmetic sequence is 23 and the 9th term is 39. Find the $n$th term [3 marks].**

Four steps take the sequence from 23 to 39, so $4d = 16$ and $d = 4$. The $n$th term is $4n + c$. When $n = 5$: $20 + c = 23$, so $c = 3$. The formula is $4n + 3$. Check the 9th term: $4 \times 9 + 3 = 39$ ✓. The first term is 7.

**Worked example 7: a geometric sequence in context**

**A rubber ball is dropped from 2 m and each bounce reaches 60% of the**

**previous height. Find the height of the 4th bounce [3 marks].**

The first bounce reaches $2 \times 0.6 = 1.2$ m. This is a geometric sequence with common ratio 0.6. The 4th bounce is $2 \times 0.6^4 = 2 \times 0.1296 = 0.2592$ m, or about 26 cm. The heights get smaller and smaller but never reach zero in theory.

## Worked example 8: quadratic sequence step by step

**Find the $n$th term of 6, 13, 24, 39 [4 marks].**

First differences: 7, 11, 15. Second differences: 4, 4. Half of 4 is 2, so the $n$th term starts $2n^2$. Compare: $2n^2$ gives 2, 8, 18, 32. Subtract from the sequence: $6 - 2 = 4$, $13 - 8 = 5$, $24 - 18 = 6$, $39 - 32 = 7$. These form an arithmetic sequence with $n$th term $n + 3$. The answer is $2n^2 + n + 3$. Check with $n = 4$: $32 + 4 + 3 = 39$ ✓.

## Note: notation

The term in position $n$ is written $T(n)$ or $u_n$. The expression $n^2$ means “the position squared”, and $2^n$ means 2 to the power of the position, which is a geometric sequence with ratio 2. Do not confuse $2n$ (even numbers) with $2^n$ (powers of two) and $n^2$ (squares).

## Watch out: the first term

When you find the $n$th term of an arithmetic sequence, the constant is not the first term. It is the first term minus the common difference. For 7, 11, 15, ... the first term is 7 but the formula is $4n + 3$, because $7 - 4 = 3$.

## Did you know?

The most famous sequence in mathematics is probably the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... It was introduced to Europe in 1202 by Leonardo of Pisa, known as Fibonacci, who used it to model the growth of a population of rabbits. The sequence appears all over nature: the number of petals on many flowers is a Fibonacci number, and the spirals of seeds in a sunflower head follow two neighbouring Fibonacci numbers. The ratio of consecutive terms gets closer and closer to the golden ratio, about 1.618. A story is told about the young Carl Friedrich Gauss, who was asked in school to add the numbers from 1 to 100 and found the answer, 5050, in seconds by pairing numbers.

## Otto's tip

Write the sequence on one line and the multiples of the common difference on the line below. The difference between the lines is the constant you need.